Double Best Response as a Network Stability Concept

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***Abstract*—We introduce a novel concept of network formation with rational players. The concept is called double best response by analogy with conventional best response dynamics used in network formation games. We applied double best response to the game called minimal-cost connectivity game where every player wants to be connected to as many players as possible with minimal individual cost. We introduce a concept of equilibrium in double best responses (EDBR) and show that the set of EDBR profiles is a subset of Nash equilibria profiles and EDBR networks demonstrate some nice properties in comparison with conventional NE.**

I. Introduction

A. Motivation

Network formation by rational self-interested players is widely studied in economy and social science [9]. Recently similar problems was investigated in the context of wireless networking.

Wireless sensor and mobile ad hoc networks consist of battery-powered devices. Efficient topology formation is a possible technique to reduce power cost and interference. Game-theoretic approach studies situations when network nodes cannot coordinate their actions and selfishly optimize their individual utilities. This can occur if nodes are controled by different selfish owners. Another case is when network nodes are produced by different vendors and there is no common protocol for efficient topology formation.

Authors in [6], [8] studied the problem of wireless network formation by rational self-interested nodes. Their approach was based on well-known iterated best response dynamics. The game have multiple equilibria and myopic best response dynamics often leads to a rather inefficient network.

B. Problem

We propose a novel game-theoretic dynamics called double best response. The main idea is that a player optimizes the utility taking into account possible future actions of other players.

Here we study double best response applied to another problem called minimal-cost connectivity game where a player wants to be connected (via paths of arbitrary length) to as many players as possible with minimal individual cost. It is supposed that a link (i, j) appears only if both players announce their willingness to be linked. A nodes choice is not their power but a set of nodes $x \in X \subseteq N$. Every action profile x induces a graph $g(x)$. We investigate properties of networks formed by players using double best response.

We consider the case when players represent wireless nodes located on a two-dimensional plane. The cost of a link between players i and j is $w(i, j) = (d_{ij})^\alpha$ where d_{ij} is the Euclidean distance and $\alpha \geq 2$ defines the strength of wireless signal attenuation.

C. Contribution

We show that there is an instance which has a Nash equilibrium with total cost $\Theta(n^\alpha)$ times the cost of the minimum spanning tree (the optimal solution). There also exist Nash equilibria where the network is disconnected (for example an empty network is NE). Next we introduce the concept of an equilibrium in double best responses (EDBR) profile and show that all EDBR networks have the following properties:

1) Connectivity. For example an empty network is not EDBR;
2) Every EDBR network is a Nash equilibrium;
3) If a network $g(x)$ is EDBR then $g(x)$ is a tree and every player i is linked to every connected component g^k of the graph $g(\emptyset, x_{-i})$ by the link (i, j^k) such that $w_{ij^k} = \min_{j \in g^k} w_{ij}$. A tree that satisfies this property was called a rational spanning tree (RST).
4) If players are located on a plane then every EDBR network is a subgraph of the Relative Neighbourhood Graph (RNG) which is a well-known structure in topology control literature [10].
5) Previous statement implies that an arbitrary EDBR network has upper bound of $O(n^\alpha)$ times the optimal. But this bound isn't proved to be tight for EDBR. Since that one can hope that EDBR actually has total cost with constant factor of an optimum.

Future research will focus on few directions. The efficiency of EDBR networks is still an open problem (are they Pareto-efficient and what is the strict bound of the total cost?). Second direction is to study computational and communication complexity of double best response compared for example with distributed algorithms for minimum spanning trees [7], [5]. It is also of our interest to investigate other network formation problems with objectives taking into account path length etc.

The paper has the following structure. Section II contain a brief survey of related results. Section III provides the framework description and properties of the game studied. Double best response concept and the main contribution are provided in the section IV. Section V concludes the paper and depicts direction of future research.

II. Related work

Wireless networks topology control is a widely studied subject [1], [16]. Game theoretic approach to ad hoc network formation was studied in [6], [8]. The authors analyzed a game where wireless nodes change their transmission range and form a link to every node inside the range. The authors proposed algorithms based on conventional best response where a player supposes the action profile is static.

In [2], [3] algorithms based on double best response were proposed for the problem studied in [8]. Node's action is the transmission power p_i and a wireless link (i, j) is formed when $p_i \geq h_{ij}$ and $p_j \geq h_{ji}$ where h_{ij} is a power threshold. Here we study a modified game where a node's action is not a transmission power but a subset of its neighbours. This slight modification allows us to analytically prove attractive properties of EDBR. Specifically in this game every EDBR network is a connected graph and every EDBR network is a subset of the relative neighbourhood graph.

In [15] a myopic game-theoretic algorithm was applied to formation of retranslation tree of next-generation (LTE or WiMAX) wireless networks. Authors of [12] studied a network formation game between primary and secondary users in a cognitive radio network. All these works studied algorithms with myopic players and we try to introduce a kind of "predictive" rationality.

Our double best response concept is based on the notions of reflexive games [13], [14]. They study players with different "rank of reflexion". A player with zero rank uses the conventional best response. Players with first rank is able to predict actions of zero-ranked players. Second-ranked players are able to predict actions of zero- and first-ranked players and so on. Similar model was called k-level and used in [4] as an explanation of experimental data on beauty-contest and stag-hunt games. In the double best response model all players can be considered as first-ranked but they are "misinformed" about the ranks of other players.

There is a lot of work on social and economic network formation [9]. Double best response doesn't directly coincide with pairwise-stability and different cooperative stability concepts. Note that players in our model still don't cooperate and payoffs are non-transferable. We believe our concept can be applied to social and economic networks. This is a direction of future research.

III. Preliminaries

A. Network formation problem

There exists a set of wireless nodes $N = \{1, \dots, n\}$. A node i can establish a link with any other node from some subset $N_i^{max} \subseteq N$. Denote action sets as $X_i = 2^{N_i^{max}}$. Node i announces its action as a subset of nodes $x_i \in X_i$. Denote $X = \prod_{i \in N} X_i$ and $X_{-i} = \prod_{j \neq i} X_j$. An action profile $x = (x_1, \dots, x_n)$ will also be referred to as a situation. Denote $x = (x_i, x_{-i})$ where $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

An action profile $x \in X$ induces an unoriented graph $g(x) = (N, E(x))$ where every node $i \in N$ is associated with a player and $(i, j) \in E(x)$ iff $i \in x_j$ and $j \in x_i$. So an edge (i, j) is formed if and only if both nodes have announced their willingness to be linked, loops (i, i) are not considered. Denote as x^{max} a profile where $x_i^{max} = N_i^{max}$ and denote $g^{max} = g(x^{max})$.

Nodes i and j are called connected in the graph $g(x)$ if $g(x)$ contains a path from i to j. Denote the set of all possible edges as $E = \{(i, j) \mid i, j \in N, i \neq j\}$. Denote the weight of an edge $e \in E$ as $w(e) \in (0, +\infty)$ or alternatively as $w(i, j)$ the weight of edge $(i, j) \in E$.

Denote as $C_i(x)$ the individual cost of node i. Consider cost $C_i(x)$ as the total power required to maintain the out-links of node i:

$$C_i(x_i) = \sum_{j \in x_i} w(i, j). \tag{1}$$

Note that if $j \in x_i$ then node i bears the cost $w(i, j)$ even if $i \notin x_j$.

The network formation problem is to choose a profile x providing a connected graph and minimizing the total cost.

$$\sum_{i \in N} C_i(x_i) \to \min \tag{2}$$

$$g(x) \text{ is connected} \tag{3}$$

Weights $w(i, j)$ depend on the environment where the network is deployed. We consider networks located on a plane and weights are defined as follows:

$$w(i, j) = (d_{ij})^{\alpha}, \tag{4}$$

where d_{ij} – Euclidean distance, $\alpha \geq 2$ – path loss exponent depends on the physical conditions. In the open air $\alpha = 2$, in an environment with obstacles $\alpha \in [2, 6]$. This model of a wireless network is common for topology control problems [16].

The problem (2) can be solved by a centralized or distributed optimization algorithm. But next we study a game-theoretic formulation where every node tries to maximize its local utility function.

B. Minimal cost connectivity game

Define a minimal cost connectivity game $\Gamma = \langle N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$:

1) A set of nodes N is a set of players;
2) $X_i = 2^{N_i^{max}}$ is an action set;
3) $u_i : X \to \mathbb{R}$ are utility functions.

Utility functions are defined as follows

$$u_i(x) = M f_i(g(x)) - C_i(x_i), \tag{5}$$

where $f_i(g(x))$ – a number of nodes reachable (via at least one path) from node i in the graph $g(x)$, $C_i(x_i)$ is cost (1) and $M > \max_{i,j \in N} w(i, j)$ – a constant which ensures the priority of connectivity over costs.

In the game-theoretic formulation we are looking not for a globally-optimal solution but for a locally-stable profile.

Definition 1: A profile $x^* \in X$ is a Nash equilibrium if $\forall i \in N$ and $\forall x_i \in X_i$

$$u_i(x_i, x^*_{-i}) \le u_i(x^*) \tag{6}$$

The goal of a player is to be connected to as many players as possible with minimal personal costs. The game is called minimal cost connectivity game and it has multiple Nash equilibria. For example an empty graph $x_i = \emptyset \ \forall i \in N$ is an equilibrium. This fact is obvious and we don't introduce it as a proposition. We are interested only in those equilibria that provide a connected graph.

Definition 2: A profile $x \in X$ is feasible if $g(x)$ is connected.

Definition 3: A profile $x \in X$ is infeasible if $g(x)$ is disconnected and there exists a profile $x' \in X$ such that $g(x')$ is connected.

Next we consider only the symmetric case where $w(i,j) = w(j,i) \ \forall i, j \in N$. Introduce a characterization for all feasible equilibria in the minimal-cost connectivity game.

Definition 4: Call a set of nodes j such that $i \in x_j$ an in-neighbourhood $A_i(x)$ of a node $i \in N$

Definition 5: Call a profile $x \in X$ a mutual situation if $\forall i \in N \ x_i = A_i(x)$ that is $i \in x_j \Leftrightarrow j \in x_i \ \forall i, j \in N$.

Proposition 1: A profile $x \in X$ is a feasible equilibrium if and only if

1) x is a mutual situation
2) $g(x)$ is a tree

Proof: The proof follows from the facts:

1) In a mutual situation a player i can not establish a new bidirectional link by adding some player j to the action x_i.
2) If $g(x)$ is a tree then deletion a link by any player i decrease $f_i(g(x))$ and therefore decrease $u_i(x)$.

■

An equilibrium is a local optimum of the problem (2) in the sense that an agent can not improve neither local utility nor total cost. Note that if x is a feasible equilibrium then $g(x)$ is a pairwise stable network [9]. In the next subsections we describe a basic algorithm of network formation called iterated best response (IBR) that obtains a feasible locally-optimal equilibrium.

C. Best response

Definition 6: Best response of a player i to a situation x is an action

$$BR_i(x) = \arg\max_{a \in X_i} u_i(a, x_{-i}). \tag{7}$$

Usually best response is considered as a function of x_{-i} but notation $BR_i(x)$ will be more convenient in the next. Best response of a player in a minimal-cost connectivity game is not necessary unique. Describe a simple algorithm that computes one of the possible best responses of a player i to a situation x. Figure 1 illustrates the algorithm.

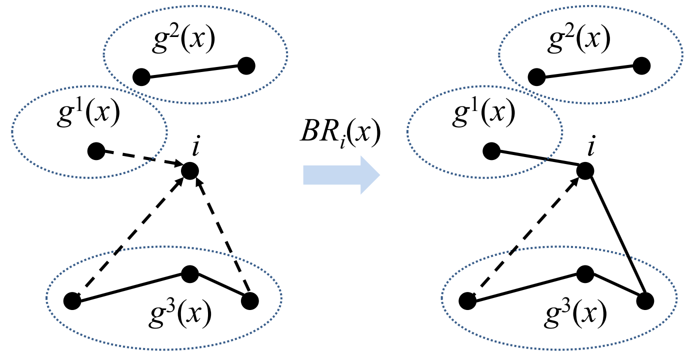

Fig. 1. Best response in a minimal-cost connectivity game.

Proposition 2: The following algorithm computes one of the possible best responses of a player i to a situation x.

1) Suppose $x_i = \emptyset$. Denote $x' = (\emptyset, x_{-i})$.
2) Suppose graph $g(x')$ contains $n(x')$ connected components: $g(x') = g^1, \ldots, g^{n(x')}$.
3) For every component g^k select a node j_k such that
$$j_k = \arg\min_{j \in g^k \cap A_i(x)} w(i,j) \tag{8}$$
If there are more than one such node then choose a node with minimal identifier.
4) $$BR_i(x) = \{j_1, \ldots, j_{n(x')}\} \tag{9}$$

This algorithm provides a node with a set of links that maximizes its utility (5) given a profile x_{-i}. Note that we define an algorithm that provide a unique best response (7) of an player i to any situation x. Next we denote as $BR_i(x)$ the action computed by the described algorithm. Denote the vector of simultaneous best responses of all agents to a situation x as $BR(x) = (BR_1(x), \ldots, BR_n(x))$.

Introduce an alternative definition of a Nash equilibrium.

Definition 7: A profile $x \in X$ is a Nash equilibrium if $\forall i \in N$
$$x_i = BR_i(x) \tag{10}$$
or equivalently
$$x = BR(x) \tag{11}$$

Consider the iterated best response algorithm.

D. Network formation process

Set a turn order for the players. Denote as i^t a player who adjust the action on step t. The iterated best responses (IBR) algorithm is defined as follows.

1) Fix initial profile x_i^0 and $g^0 = g(x^0)$.
2) On a step t choose a player i who updates the action:
$$x_i^{t+1} = BR_i(x^t), \ x_j^{t+1} = x_j^t, \ j \neq i \tag{12}$$

3) Update network

$$g^{t+1} = g(x^t) \tag{13}$$

Then go to step 2.

4) The process stops when no player changes the action. Formally, $\forall i \in N \ x_i^t = BR_i(x^t)$

This procedure is well known. It's easy to show that iterated best response converges to a Nash equilibrium and the process lasts exactly one round e.g. n action updates. This procedure was studied in [6], [8] as an algorithm of wireless ad hoc network formation.

The properties of iterated best response dynamics can be summarized as follows:

1) Iterated best response converges to a Nash equilibrium. It takes exactly n steps of equation (12).
2) If $g(x^0)$ is connected then $g(x^t)$ is connected for $t = 1, 2, \ldots$. For example if $x^0 = x^{max}$ and g^{max} is connected.

Next we show that the resulting equilibria can be significantly inefficient. In the section IV we propose a more sophisticated concept to improve the performance of the network formation algorithm.

E. Efficiency of Nash equilibria

Here we analyze equilibria that can be obtained by IBR algorithm. So we consider only feasible equilibria. We also focus our study on the case of symmetric link weights $w(i,j) = w(j,i)$. Note that in this case a minimum spanning tree (MST) is the globally optimal solution of the problem (2).

Definition 8: The price of anarchy (PoA) of a minimal-cost connectivity game Γ is the relation

$$PoA(\Gamma) = \frac{C(x^{worst})}{C(MST)}, \tag{14}$$

where $C(x^{worst})$ – the total cost (2) of the worst feasible equilibrium and $C(MST)$ – the total cost of the minimum spanning tree that is the optimal solution.

Lemma 1: If link weights are symmetric i.e. $w(i,j) = w(j,i) \ \forall i,j \in N$ then the optimal solution of the problem (2) is a minimum spanning tree (MST) of the graph g^{max}.

Proof: This lemma follows from the definition of MST and the fact that if $w(i,j) = w(j,i)$ then $\sum_{i \in N} w_i(x_i) = 2\sum_{e \in E} w(e)$ ∎

Since any tree is a NE we see that the worst NE is a maximum spanning tree of the graph g^{max}. If weights $w(i,j)$ are arbitrary then the price of anarchy can be arbitrary large. Consider the price of anarchy for a network located on the Euclidean plane.

Proposition 3: If link weights are $w_{ij} = (d_{ij})^\alpha$ where d_{ij} is a Euclidean distance between nodes i and j then there exists a minimal-cost connectivity game Γ such that

$$PoA(\Gamma) = \Theta(n^\alpha) \tag{15}$$

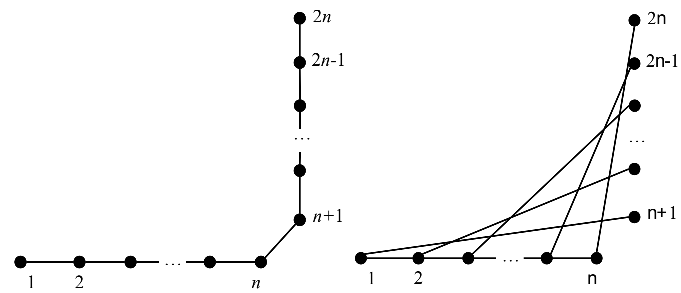

Fig. 2. Minimal-cost connectivity game with $PoA = \Theta(n^\alpha)$. Left – MST. Right – an equilibrium with $C(x) = \Theta(n^\alpha)C(MST)$.

Proof: Figure 2 shows such a game (call it Γ_{tr} – triangle). A distance between any two adjacent nodes i, $i+1$ equals $d_{ii+1} = 1$.

1) Figure 2a shows the MST. The cost of MST is linear $C(MST) = 2n - 1$.
2) Total cost of any Nash equilibrium is bounded as $O(n^\alpha)C(MST)$. Prove it. Any possible link has the cost no more than $w_{max} \approx (\sqrt{2}(n-1))^\alpha$. Any tree with $2n$ nodes contains $2n-1$ edges therefore $\forall x \in X$ such that x is a Nash equilibrium $C(x) \le (2n-1)w_{max} = (2n-1)(\sqrt{2})^\alpha(n-1)^\alpha = O(n^\alpha)C(MST)$.
3) Figure 2b shows the equilibrium with $C(x) = \Theta(n^\alpha)C(MST)$. The minimal link cost is $w_{min} \approx (\frac{\sqrt{2}}{2}\frac{n}{2})^\alpha$. The maximal link cost is $w_{max} \approx n^\alpha$. Therefore $(2n-1)(\frac{\sqrt{2}}{2}\frac{n}{2})^\alpha \le C(x) \le (2n-1)n^\alpha$ and $(\frac{\sqrt{2}}{4})^\alpha n^\alpha C(MST) \le C(x) \le n^\alpha C(MST)$. That is $C(x) = \Theta(n^\alpha)C(MST)$ and $PoA(\Gamma) = \Theta(n^\alpha)C(MST)$.

∎

We found that conventional best response can obtain an equilibrium that is polynomially worse than the global optimum. In the next section we consider a novel game dynamics called double best response which seems to form more efficient networks.

IV. Double best response

A. General description

If a player use best response (7) the profile x_{-i} is supposed to be static in the future. We propose a decision model where a player consider possible reactions of the opponents to her choice. Denote as $BR_S(x)$ a vector of simultaneous best responses of agents $j \in S \subseteq N$ to a profile x.

Definition 9: Double best response of a player i to a profile x is the action

$$DBR_i(x) = arg \max_{a \in X_i} u_i(a, BR_{-i}(a, x_{-i})) \tag{16}$$

where

$$BR_{-i}(a, x_{-i}) = (BR_1(a, x_{-i}), \ldots, BR_{i-1}(a, x_{-i}), BR_{i+1}(a, x_{-i}), \ldots, BR_n(a, x_{-i}))$$

Here $BR_{-i}(a, x_{-i})$ are simultaneous best responses of players $j \neq i$ to the new profile where $x_i = a$.

Double best response can be viewed as the following procedure. A player i chooses an action $a \in X_i$. Then she computes best responses of other players $BR_{-i}(a, x_{-i})$ and the payoff $u_i(a, BR_{-i}(a, x_{-i}))$. If the action space X_i is finite then double best response can be found by brute-force search through action space. Further we show that in the minimal-cost connectivity game double best response is transformed to a natural local decision rule.

The equation (16) defines a "global" double best response rule where a player is able to compute best responses of any other player in the network. That seems too strong for large networks and we propose also a "local" modification.

Definition 10: Reflexive set $R_i \subseteq N \setminus \{i\}$ of a player $i \in N$ is the set of players $j \neq i$ such that i is able to compute (or get in another way) best response $BR_j(x)$.

Definition 11: Local double best response of a player i with reflexive set R_i to a profile x is the action

$$DBR_{i,R_i}(x) = \arg\max_{a \in X_i} u_i(a, x_{N \setminus R_i}, BR_{R_i}(a, x_{-i})). \quad (17)$$

Here $x_{N \setminus R_i}$ denotes actions x_j of all players $j \notin R_i$ and $BR_{R_i}(a, x_{-i})$ – best responses of all players $j \in R_i$ to the profile (a, x_{-i})

In the limit case $R_i = \emptyset$ local double best response transforms to the myopic best response 7. A natural choice is $R_i = X_i$ where X_i contains all nodes to those i is able to connect. In the case of wireless network this means that a node can compute best responses of other nodes located in its maximal transmitting range.

B. Equilibrium in double best responses

Double best response can be viewed as a dynamic decision rule. Here we study situations that remain stable when players use double best response.

Definition 12: Call a profile x an Equilibrium in Double Best responses (EDBR) if $\forall i \in N$

$$x_i = DBR_i(x). \quad (18)$$

Characterize some properties of EDBR profiles for a minimal-cost connectivity game. Here we focus on global modification of double best response and suppose that g^{max} is connected to clarify the main results. In fact here we consider the case when $X_i = 2^N$. All of the results remain true when local double best response is used with $R_i = X_i$ and $X_i \subset 2^N$ (and g^{max} is connected). Note that symmetry $w(i,j) = w(j,i)$ is essential.

Denote $x_i^{dbr} = DBR_i(x)$ and $K_i(x)$ – connected component of a node i in the graph $g(x)$. Denote $i \sim j$ if i and j are connected in the graph $g(x)$ and $i \nsim j$ if i and j are not connected.

First introduce few lemmas. Their proofs are omitted because of their simplicity.

Lemma 2: $\forall x \in X$ $g(BR(x)) \subseteq g(x)$ i.e. simultaneous best response doesn't form new links.

Lemma 3: $\forall i, j, k \in N$ if $\exists y \in X_i$ and $(j,k) \in g(y, BR_{-i}(y, x_{-i}))$ then $\forall x_i \in X_i (j,k) \in g(x_i, x_{-i})$.

Lemma 4: Graph $g(BR(x))$ doesn't contain cycles. And $\forall j,k \in N$ such that $j \sim k$ in $g(BR(x))$ if $(j,k) \notin g(BR(x))$ then $j \notin BR_k(x)$ and $k \notin BR_j(x)$.

Lemma 5: If $j \in DBR_i(x)$ then $i \in BR_j(DBR_i(x), x_{-i})$

Proposition 4: If x is EDBR then x is feasible i.e. $g(x)$ is connected.

Proof: By contradiction suppose $\forall i \in N$ $x_i = DBR_i(x)$ and $g(x)$ is disconnected. Choose two components g^1, g^2. Then choose a pair of nodes $i \in g^1$ and $j \in g^2$ such that $w_{ij} = \min_{i \in g^1, j \in g^2} w_{ij}$. As i and j belong to different connected components $i \notin x_j$ or $j \notin x_i$. Suppose $j \notin x_i$. Suppose $x_i' = x_i \cup \{j\}$ and prove that $u_i(x_i', BR_{-i}(x_i', x_{-i}) > u_i(x_i, BR_{-i}(x_i, x_{-i}))$. Best responses of nodes $k \neq i \neq j$ will not change and best response of j now contains i. Then $f_i(x_i', BR_{-i}(x_i', x_{-i}) > f_i(x_i, BR_{-i}(x_i, x_{-i}))$ and $u_i(x_i', BR_{-i}(x_i', x_{-i}) > u_i(x_i, BR_{-i}(x_i, x_{-i}))$ that is impossible if $x_i = DBR_i(x)$. ■

Proposition 4 means that double best response eliminates all infeasible equilibria. Note that if conventional best response starts from infeasible solution it may not restore the connectivity.

Proposition 5: If x is EDBR then $g(x)$ doesn't contain cycles.

Proof:

Suppose $g(x)$ contain a cycle $c = (i_1, \ldots, i_r, i_1)$. Then by lemma 5 and condition $x = DBR(x)$ we have $i_k \in BR_{i_{k+1}}(x)$ and $i_{k+1} \in BR_{i_k}(x)$, $k = 1, \ldots, r$,$i_1 \in x_{i_r}$, $i_r \in x_{i_1}$. It means that $g(BR(x))$ contain a cycle that is a contradiction to lemma 4. ■

Corollary 1: If x is EDBR then $g(x)$ is a spanning tree of g^{max}

Proof: By proposition 4 $g(x)$ is connected and by proposition 5 $g(x)$ doesn't contain cycles. Therefore $g(x)$ is a spanning tree of g^{max} (note g^{max} is supposed to be connected). ■

Corollary 2: If x is EDBR then x is a mutual situation.

Proof:

Suppose there exist $i, j \in N$ such that $i \in x_j, j \notin x_i$. Since $g(x)$ is connected by proposition 4 there exists a path $p = (j = i_1, i_2, \ldots, i_r = i)$ therefore $\exists i_2 \in p$ such that $j \in x_{i_2}$. By lemma 5 if $j \in x_{i_2}$ and $j \in x_i$ then $i, i_2 \in BR_j(x)$. But i_2 and i should be a member of the same component of the graph $g(\emptyset, x_{-i})$ that is a contradiction to the proposition 2. ■

Proposition 6: If x is EDBR then x is a Nash equilibrium.

Proof: This directly follows from proposition 1 and corollaries 1,2. ■

We have just shown that the set of EDBR profiles is a subset of feasible Nash equilibria. Next proposition gives a more strict characterization of EDBR profiles.

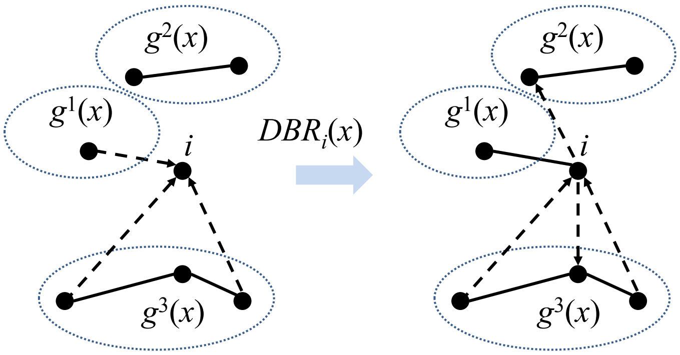

Fig. 3. Double best response in a minimal-cost connectivity game.

Proposition 7: If x is EDBR then $\forall i \in N\ x_i = \{j_1, \ldots, j_{n(x')}\}$ where

$$j_k \in \arg\min_{j \in g^k} w(i,j) \tag{19}$$

where g^k is a connected component of the graph $g(x')$, $x' = (\emptyset, x_{-i})$.

Proof: Consider graph $g(\emptyset, x_{-i}) = g^1 \cup \ldots \cup g^{n(x')}$. Choose a node $i \in g^1$. As noted in the proof of proposition 4 there exists a node $j \in g^2$ such that $w(i,j) = min_{k \in g^2} w(i,k)$. If there are more than one such node we resolve ties ordering them by numbers. There exist a node $k \in g^2$ such that $k \in x_i^{dbr}$. If $j \neq k$ then $w(i,j) = w(i,k)$. If not then node i can improve its utility $u_i(x_i^{dbr}, BR_{-i}(x))$ by connecting to j and disconnecting from k. So the condition 19 holds if x is EDBR. ∎

The difference between arbitrary Nash equilibrium and EDBR situation is as follows. When node i computes its best response it searches over only those components where at least one node gives it an "offer". In a EDBR profile a node can expect the positive reaction from a neighbour even if there is no "offer" from this neighbour. Note that condition (19) implies that for minimal-cost connectivity game double best response does not require additional information compared with conventional best response. Figure 3 illustrates the concept.

The condition (19) introduce a special class of trees. Call them rational spanning trees.

Definition 13: If a tree satisfies the condition(19) then it is a rational spanning tree – RST.

The notion of RST can be explained as follows. Consider a tree T and a node $i \in T$. If node i remove all its incident edges then T will split into several subtrees. Then suppose node i can choose a set of edges to be connected with all the subtrees with minimal total edge cost. It is rational for i to choose the edges according to the condition (19).

The notion of RST does not require game-theoretic framework. The characterization of EDBR profiles for minimal-cost connectivity game as a rational spanning trees allows us to establish the existence of EDBR.

The set of RST is not empty. For example a minimal spanning tree is also a rational spanning tree.

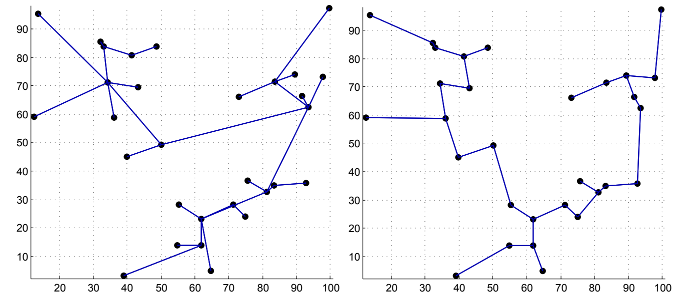

Fig. 4. Left – a Nash equilibrium network. Right – an EDBR network for the same location.

Proposition 8: A MST is also a RST.

Proof: By contradiction. Let $g^{mst} = (N, E^{mst})$ is MST and is not RST. Then there exists a node $i \in N$ and a component g^k such that $j_k \notin \arg\min_{j \in g^k} w(i,j)$. There must exists a node $j'_k \in g^k$ such that $(i, j'_k) \notin E^{mst}$ and $w(i, j'_k) < w(i, j_k)$. Then we can reduce the total cost by replacing link (i, j_k) by (i, j'_k). The contradiction as g^{mst} is a minimal spanning tree. ∎

Corollary 3: For a minimal-cost connectivity game there exists at least one EDBR profile.

Proof: MST already exists and MST is a EDBR. ∎

We can conclude that for a minimal-cost connectivity game the set of EDBR profiles is a subset of feasible Nash profiles and there exists at least one EDBR profile. Double best response can be viewed as a distributed algorithm for computing a rational spanning tree. Next subsection provides some findings on the geometric properties and efficiency of EDBR.

C. Efficiency of equilibria in double best responses

Next we analyze the efficiency of EDBR profiles. The results above hold for any symmetric weights $w(i,j)$. Here we study properties of double best response if nodes are located on a 2d Euclidean plane. We took the concept of Relative Neighbourhood Graph (RNG) proposed in computational geometry [17].

Definition 14: Relative Neighbourhood Graph (RNG) over nodes N is an unoriented graph $RNG(N) = (N, E^{RNG})$ such that $(i,j) \in E^{RNG}$ iff $\forall k \in N,\ k \neq i,j$ the distance $d_{ij} \leq \max\{d_{ik}, d_{jk}\}$.

Proposition 9: If x is EDBR then $g(x) \subseteq RNG(N)$.

Proof: This follows by contradiction from proposition 7. Suppose $g(x) \nsubseteq RNG(N)$. Then $\exists i,j \in N$ such that $(i,j) \in g(x)$ and $\exists k \in N$ such that $d_{ij} > \max\{d_{ik}, d_{jk}\}$. Apply proposition 7 to node i. Suppose $k \sim j$ in the graph $g(\emptyset, x_{-i})$. The condition $d_{ij} > \max\{d_{ik}, d_{jk}\}$ contradicts to the proposition. If $j \nsim k$ in $g(\emptyset, x_{-i})$ then the same situation holds when apply the proposition to node k and $i \sim j$ in the graph $g(\emptyset, x_{-k})$. ∎

This proposition gives a top bound on the total cost of EDBR networks.

Proposition 10: If x is a EDBR then $g(x)$ has total cost $C(x) \leq O(n^\alpha)C(MST)$.

Proof: It is known that total cost of RNG has a strict top bound of $O(n^\alpha)$ times of the optimum [11]. ■

Structures based on RNG are widely used in topology control and RNG is considered as an efficient structure for wireless network topology [10]. The fact that a game-theoretic rule leads to a concept from computational geometry was not evident. Figure 4 shows an arbitrary Nash equilibrium network and an EDBR network.

V. CONCLUSION

We studied a novel network formation concept called double best response. The concept was analyzed in the framework of a so-called minimal-cost connectivity game. Every player would like to be connected to as many players as possible and minimize the individual cost. In the special case players correspond to wireless nodes located on a plane. The game was shown to have multiple Nash equilibria. There are so-called infeasible Nash equilibria where the network is not connected (like empty network). Even considering only feasible equilibria the game still has polynomial price of anarchy.

Double best response of a player is an action that maximizes the utility under assumptions that the player is able to predict the best responses of other players to that action. Therefore dynamics based on double best responses can be called "predictive". By analogy with Nash equilibrium we introduce a concept of Equilibrium in Double best responses (EDBR). We established the existence of EDBR and showed that in this game the set of EDBR profiles is a subset of feasible Nash equilibria profiles.

We found that EDBR networks is trees characterized by a local property that in some sense is more efficient than conventional Nash equilibrium. We denote this class of trees as rational spanning trees (RST). Also we showed that every EDBR network (called now RST) is a subset of the Relative Neighbourhood Graph (RNG). Relative Neighbourhood Graphs are widely used in computational geometry and wireless network topology control. The top bound (not tight) on the total cost of EDBR profiles is $O(n^\alpha)$ where $\alpha \geq 2$ is the path-loss exponent.

Future research will focus on the efficiency of EDBR networks. The open questions is are they Pareto-efficient and does EDBR guarantee total cost better than the worst Nash equilibrium. The total cost of a Nash equilibrium in the studied game has tight top bound of $O(n^\alpha)$ times the optimum. Since this bound for EDBR is not tight one can hope it can be improved to $O(1)$ times the optimum.

Second direction of future investigation will be algorithmic complexity of double best response in a more general framework. For the game studied here double best response require local link weights knowledge. It is an open question what information is required for double best response in general and what the computational complexity is.

We also believe that double best response concept can be applied to other network formation problems.

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