Minimization of maximum lateness with equal processing times for single machine

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Abstract: The following case of the classical NP-hard scheduling problem is considered. There is a set of jobs \( N \) with identical processing times \( p = \text{const} \). All jobs have to be processed on a single machine. The objective function is minimization of maximum lateness. We analyze algorithms for the makespan problem, presented by Garey et al. (1981) and Simons (1978) and represent two polynomial algorithms to solve the problem and to construct the Pareto set with respect to criteria \( L_{\text{max}} \) and \( C_{\text{max}} \). The complexity of presented algorithms equals \( O(Q \cdot n \log n) \) and \( O(n^2 \log n) \), where \( 10^{-Q} \) is the accuracy of the input-output parameters.

Keywords: scheduling, unit-time jobs, polynomial algorithms, dynamic programming

1. INTRODUCTION

1.1 Formulation of the main problem

The following problem of scheduling theory is considered. There is a set of jobs \( N \) and a single machine to process jobs from this set. For each job \( j \in N \), a release date \( r_j \) and a due date \( d_j \) are given. The processing time \( p \) is the same for all jobs of the set \( N \). We define a schedule \( \pi \) as an execution sequence \( K_1(\pi), K_2(\pi), \ldots, K_n(\pi) \), where \( K_i(\pi) = j \) means that job \( j \in N \) is processed under the ordinal number \( i \) under the schedule \( \pi \). The execution of the job \( K_i(\pi) = j \) starts at time \( R_i(\pi) = \max(C_{i-1}(\pi), r_K(\pi)) \) and finishes at time \( R_i(\pi) + p = C_j(\pi) \), where \( C_j(\pi) \) is the completion time of the job \( j \in N \). Let us denote lateness as \( L_j(\pi) = C_j(\pi) - d_j \). The maximum completion time and maximum lateness are denoted as \( C_{\text{max}} \) and \( L_{\text{max}} \) respectively. Let us call the schedule \( \pi \) allowable for the set \( N \) if all jobs under the schedule \( \pi \) executed without preemptions and intersections. We denote the set of all allowable schedules as \( \Pi \). The goal is to find allowable schedule \( \pi \in \Pi \), which satisfies the following optimization criteria:

\[
\min_{\pi \in \Pi} \max_{j \in N} L_j(\pi).
\]

This problem \( 1|r_j,p_j = p|L_{\text{max}} \) is a special case of classical NP-hard scheduling problem \( 1|r_j|L_{\text{max}} \). Now, let us consider some approaches to obtain the solution in polynomial time.

A simple way to obtain the solution is the dichotomy (trisection search) method. In the first step, we find boundary values on the objective function \( L_{\text{max}} \). Each job \( j \in N \) holds:

\[
d_j + L_j = C_j \geq r_j + p.
\]

Hence, a lower bound \( L_{B_0} \) on the optimal value \( L_{\text{max}} \) is as follows:

\[
L_{B_0} = \min_{j \in N} \{r_j - d_j\} + p.
\]

An upper bound \( U_{B_0} \) can be estimated as:

\[
U_{B_0} = L_{\text{max}}(\pi_C),
\]

where \( \pi_C \) is an optimal schedule for the problem \( 1|r_j,p = \text{const}|C_{\text{max}} \). The fastest algorithm for solving this problem was presented by Garey et al. (1981). Let us use this algorithm to construct \( \pi_C \) in \( O(n \log n) \) operations.

After finding the bounds \( L_{B_0} \) and \( U_{B_0} \) we use dichotomy method to find a solution of the problem as follows. In the first step we divide the interval \([L_{B_0}, U_{B_0}]\) on three parts. Then, we set deadlines for all \( j \in N \):

\[
\begin{align*}
d'_j &= d_j + \frac{2}{3} L_{B_0} + \frac{1}{3} U_{B_0}; \\
d''_j &= d_j + \frac{1}{3} L_{B_0} + \frac{2}{3} U_{B_0}.
\end{align*}
\]

Then we use algorithm presented by Garey et al. (1981) to construct the optimal schedules \( \pi_{1C} \) and \( \pi_{2C} \) for the problems \( 1|r_j,d'_j,p = \text{const}|C_{\text{max}} \) and \( 1|r_j,d''_j,p = \text{const}|C_{\text{max}} \) respectively. If the schedule \( \pi_C \) exists, set:

\[
\begin{align*}
L_{B_1} &= L_{B_0} \\
U_{B_1} &= \frac{2L_{B_0} + U_{B_0}}{3}.
\end{align*}
\]

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Fig. 1. An example of the use of three times dichotomy.

If the schedule \( \pi_1 \) does not exist and the schedule \( \pi_2 \) exists, let us set:

\[
\begin{align*}
    LB_1 & := \frac{2LB_0 + 1UB_0}{3} \\
    UB_1 & := \frac{1LB_0 + 2UB_0}{3}
\end{align*}
\]

Otherwise, we set:

\[
\begin{align*}
    LB_1 & := \frac{1LB_0 + 2UB_0}{3} \\
    UB_1 & := UB_0
\end{align*}
\]

This procedure repeated until the difference \( UB - LB \) is not larger than \( 10^{-Q} \) – the accuracy of the input-output parameters. An example of the dichotomy method usage is shown in figure 1.

The number of steps is equal to

\[
\log_3((UB_0 - LB_0) \cdot 10^Q)
\]

In each step two schedules \( \pi_1, \pi_2 \) are constructed. Hence, the total complexity equals:

\[
\log_3((UB_0 - LB_0) \cdot 10^Q) \cdot 2O(n \log n) = O(Q \cdot n \log n)
\]

3. THE AUXILIARY PROBLEM

3.1 Formulation of auxiliary problem and algorithm

Let us consider the second approach. We have to formulate an auxiliary problem to construct the second algorithm. We consider the same set of jobs \( N = \{1, \ldots, n\} \) and a bound on the maximum lateness \( y \). The goal is to construct a schedule with respect to criterion:

\[
\min_{\pi \in \mathcal{R}} \max_{j \in N} C_j(\pi) / L_{\max}(\pi) < y.
\]

For each set of due dates \( d_1, \ldots, d_n \) and the bound on the lateness \( y \) deadlines \( D_j \) can be calculated by the following formula:

\[
D_j = d_j + y.
\]

The auxiliary problem is the same as problem 1\(|r, p = \text{const}|C_{\max}\), but with one exception: the completion time \( C_j \) of the job \( j \) may not exceed the deadline \( D_j \):

\[
C_j < D_j.
\]

An allowable schedule satisfying this restriction is called feasible. To construct the solution of the auxiliary problem, we consider the approach presented by Simons (1978). Next, we briefly recall the main idea and the important notations from this paper.

The algorithm works as follows. While the completion times of all jobs are not larger than their deadlines, schedule jobs according to algorithm, presented by Schrage (1970). If for any job \( X \in N \) the inequality

\[
C_X \geq D_X
\]

holds then, execute the special procedure CRISIS(\( X \)). This procedure finds the job \( A \), which is already scheduled with the latest completion time, but for which

\[
D_A > D_X
\]

holds. This job is called Pull(\( X \)), the time moment when its execution starts is denoted as \( t_{A,X} \) and all jobs which are already scheduled after \( Pull(X) \) and \( X \) constitute the restricted set \( r.s. \) \( S(A,X) \) (see fig. 2). The set of jobs, which belong to \( S(A,X) \) and not belong to any restricted subset of \( S(A,X) \) is denoted as \( S(A,X) \). The procedure CRISIS(\( X \)) reschedule a set of jobs \( \{A\} \cup S(A,X) \). The procedure fails when \( Pull(X) \) for a crisis job \( X \) does not exist. After a successful execution of the procedure CRISIS(\( X \)), Schrage’s algorithm (see Schrage (1970)) is used to schedule the jobs. Such a scheduling repeats until any call of procedure CRISIS() fails or all jobs from the set \( N \) have been successfully scheduled.

3.2 Procedures for auxiliary algorithm

This algorithm consists of the following procedures: the algorithm presented by Schrage (1970) as well as the procedures CRISIS(\( X \)) and INVASION(\( S(C,W), r_s(C,W) \)) presented by Simons (1978).

3.3 Algorithm for auxiliary problem

The solution of auxiliary problem \( 1\{|r, p = \text{const}, D_j|C_{\max}\} \), presented in Simons (1978) is as follows. Find the schedule \( \pi \) by means of Schrage’s algorithm, and use then the auxiliary algorithm.

Theorem 1. After the execution of the auxiliary algorithm, an optimal set with respect to the criterion \( C_{\max} \) is constructed, provided that the schedule \( \pi \) is constructed by Schrage’s algorithm.

Proof. The proof of this theorem is given in Simons (1978).

Theorem 2. Let \( S(A,X) \) be a restricted set. If a feasible schedule exists, then assertions 1-4 hold for all feasible schedules. Each time a procedure CRISIS() or INVASION() is about to schedule \( S(A,X) \) assertions 1-3 hold:

Fig. 2. Job X experiences crisis.

Schrage’s Algorithm

1. Find earliest release time:

\[
t = \min_{j \in N} r_j.
\]

2. Find a non-processed job, which released at the moment \( t \):

\[
i = \arg \min_{j, r_j \leq t} D_j.
\]

3. Process job \( i \), add it to \( \pi \) and remove it from \( N \):

\[
C_i = t + p; \\
\pi := \{\pi, i\}; \\
N := N \setminus \{i\}.
\]

4. Else:

\[
\text{return}(\pi).
\]
CRISIS(X)
1. Assume that X belongs to a minimal r.s. $S'$. If X does not belong to any restricted set, then $S' = N$.
Backtrack over the first level jobs of $S'$ looking for Pull(X). Let $A = \text{Pull}(X)$ and define $S(A, X)$ to be a restricted set. If no Pull(X) exists, report failure and halt.
2. Count the number of jobs of $S(A, X)$ in each first level interval of $S(A, X)$. Increase the count of the initial first level interval by 1.
3. Remove the jobs $S(a, x)$ from the schedule.
4. $i := 1$.
5. While the required number of jobs of $S(A, X)$ have not been scheduled in the $i^{th}$ first level interval, schedule the jobs of $S(A, X)$ using the naive algorithm. (If $i = 1$, the first level interval begins at $r_{S(A, X)}$; otherwise, the interval begins at the time at which the preceding r.s. is completed).
   a) If some job $Z$ has a crisis, call CRISIS(Z).
   b) If some job $Y$ invades the following r.s. $S(C, W)$, set $r_{S(C, W)}$ to be the time at which $Y$ is completed and call INVASION($S(C, W), r_{S(C, W)}$).
6. If all the jobs of $S(A, X)$ have been scheduled then return; otherwise $i := i + 1$.
7. Go to step 5.

INVASION($S(C, W), r_{S(C, W)}$)
1. Count the number of jobs of $S(A, X)$ in each first level interval of $S(A, X)$.
2. Steps 2-6 are identical to steps 3-7 of the CRISIS subroutine.

AUXILIARY ALGORITHM
1. While $N$ has not been completely checked under $\pi$, check all jobs $j \in N$ holds $C_j(\pi) < D_j$; otherwise halt ($N$ has been successfully scheduled).
   a) If some job $X$ has a crisis, call CRISIS(X).
   b) Else return($\pi$).
1. The first job of $S(A, X)$ is always scheduled to begin in $(t_{AX}, t_{AX} + p)$.
2. Only jobs in $S(A, X)$ can be scheduled totally in $(t_{AX}, D_x)$.
3. $S(A, X)$ can not be scheduled to begin before $r_{S(A, X)}$.
   When the program returns from a procedure call the following assertion holds:
4. $S(A, X)$ can not be completed any earlier than the time at which it is currently scheduled to be completed.

Proof. The proof of this theorem is given in Simons (1978).

4. MAIN PROBLEM SOLUTION

Now let us consider the main problem 1) $r_j, p = \text{const}|L_j|$. We present an algorithm to obtain the Pareto set of schedules with respect to criteria $L_{\text{max}}$ and $C_{\text{max}}$. First, we introduce a procedure CHECK($\pi, N, y$) which is as follows.

Lemma 1. Let $\pi$ and $\pi'$ be the schedules, constructed by the auxiliary algorithm for the bounds $y$ and $y'$, respectively, and

$$\pi^* = \text{CHECK}($$ $\pi, N, y$. $$ $$

CHECK($\pi, N, y$)
1. Set the bound $y$.
2. Set deadlines $D_i = d_i + y$.
3. If all jobs from $N$ have been scheduled, go to step 7.
4. While $t$ is not in the interval $[r_{S(A, X)}, d_x]$ for any restricted set $S(A, X)$ from the schedule $\pi$, execute the jobs under $\pi^*$ according to Schrage’s algorithm.
5. Otherwise, execute under $\pi$ only the jobs from the set $S(A, X)$, and then go to step 3.
6. If in steps 4-5 any job $Y$ experiences a crisis, run CRISIS(Y).
7. return($\pi^*$).

Proof. Assume the contrary. We compare the schedules $\text{CHECK(} \pi, N, y \text{)}$ and $\pi'$ from $t = 0$ up to $C_{\text{max}}$. Suppose, that at the moment $t$ the first difference was found. The possible cases of the difference illustrated in figure 3, and they are as follows:

1) Two different jobs start at the moment $t$. This type of difference is impossible, because both algorithms ensure that at the moment $t$ only the job with a minimal due date can start its execution.
2) There is an execution of the job $X$ under the schedule $\pi^*$ and an idleness under the schedule $\pi'$ at the moment $t$. The job X is not executed in the sequence $\pi'$ in spite of $r_X \leq t$. Hence, $X = \text{Pull}(Y)$ under the schedule $\pi$. According to the time $r_{S(X, Y)}$ and Theorem 2 the jobs from the set $S(X, Y)$ can not finish their execution in the schedule $\pi^*$ before the time $D_Y(\pi')$ because the first of them starts at time $t_{AX} + p$. Hence, some jobs from $S(X, Y)$ experience a crisis under the schedule $\pi^*$. In the last call of CRISIS($Y'$), $Y' \in S(X, Y)$ we have CRISIS($Y'$) = $X$. Thus the job X cannot start at the moment $t$.
3) There is an execution of the job $X$ under the schedule $\pi'$ and an idleness under the schedule $\pi^*$ at the moment $t$. The idleness is related to some restricted set $S(A, Y)$ from the set of jobs in $\pi$. The schedule $\pi'$ is feasible for the set of jobs $N$ and the deadlines $D_i(\pi^*)$ since $y < y'$. According to Theorem 2, the jobs from the set $S(A, Y)$ can not start their execution under the
MAIN ALGORITHM
1. Set the bound $y_0 := +\infty$.
2. Construct the schedule $\pi_1$ according to the auxiliary algorithm, and add it to $\Phi$, i.e.: $\Phi := \{\pi_1\}$; set the counter $k := 1$; set the bound $y_1 := L_{max}(\pi_1)$.
3. Construct the schedule $\pi_{k+1} = \text{CHECK}(\pi_k, N, y_k)$.
   a) If the schedule $\text{CHECK}(\pi_k, N, y)$ exists, then:
      * add $\pi_{k+1}$ to the set $\Phi$, i.e.: $\Phi := \Phi \cup \pi_k$;
      * set $y_k = L_{max}(\pi_k)$;
      * increase the counter $k := k + 1$;
      * repeat step 3.
   b) Otherwise, return($\Phi$).

   schedule $\pi'$ at the moment $t_{A,Y} + p$ and finish their execution until the moment $D_Y(\pi')$.

Hence, all above three cases are impossible and there are no differences between $\pi^*$ and $\pi'$.
Q.E.D.

Next, we describe the main algorithm to obtain the Pareto set with respect to criteria $L_{max}$ and $C_{max}$.

Lemma 2. If any job becomes a crisis job for the second time, then algorithm stops.

Proof. When a r.s. is formed, all the jobs in this set have deadlines no greater than the deadline of the crisis job which triggered the r.s. Consequently, if that job experiences a crisis for a second time, the algorithm will not find a job to pull and will be a failure in step 1 of the subroutine $\text{CRISIS()}$.

Theorem 3. After the execution of algorithm 4, the Pareto set of schedules $\Phi$, $|\Phi| \leq n + 1$ according to the criteria $L_{max}$ and $C_{max}$ has been constructed, and the schedule $\pi^*$ is an optimal solution for the main problem.

Proof. When Algorithm 4 has terminated, the set of schedules $\Phi$ has been constructed. For each pair of consecutive schedules $\pi_x, \pi_{x+1}$ of the set $\Phi$ the inequalities
\[
\begin{cases}
L_{max}(\pi_{x+1}) < y_{x+1}, \\
L_{max}(\pi_x) < y_x,
\end{cases}
\]
hold. Moreover, for this two schedules the inequality
\[
C_{max}(\pi_{x+1}) \geq L_{max}(\pi_x)
\]
holds, because $\pi_x$ is an optimal schedule with respect to criterion $C_{max}$ if $L_{max} < y_x$ and $y_{x+1} < y_x$. Hence,
\[
L_{max}(\pi_{|\Phi|-1}) \leq \ldots \leq L_{max}(\pi_1) < L_{max}(\pi_0),
\]
\[
C_{max}(\pi_{|\Phi|-1}) \geq \ldots \geq C_{max}(\pi_1) \geq C_{max}(\pi_0).
\]
This implies that the schedule $\pi^* = \pi_{|\Phi|-1}$ is an optimal according to the criterion $L_{max}$. On each $\text{CHECK()}$ procedure of algorithm 4 $\text{CRISIS()}$ subroutine executes at least once, because the schedules $\pi_k$ and $\pi_{k+1}$ are different. Hence, we get
\[
|\Phi| \leq n + 1.
\]

Lemma 3. The complexity of Algorithm 4 is $O(n^2 \log n)$.

Proof. According to Theorem 1, for each job $X \in N$ the procedure $\text{CRISIS}(X)$ is executed not more than once. Hence, the total number of running $\text{CRISIS()}$ is not more than $n$. During the procedure $\text{CRISIS}(X)$ each job from the set $N$ rescheduled not more than once. Hence, each job schedules not more than $n+1$ times by Schrage’s algorithm during the construction of sets $\pi_1, \ldots, \pi_{|\Phi|}$, and not more than $n$ times due to execution of $\text{CRISIS()}$ procedures. Hence, total number of reschedulings is not more than $(2n + 1) \cdot n$. It is still necessary to multiply this result on log $n$ due to the use of the heaps. This leads to a total complexity of $O(n^2 \log n)$.

5. CONCLUSION

Two approaches to solve the problem $1|r_j,p = \text{const}|L_{max}$ are presented. In addition, the Pareto set with respect to the criteria $L_{max}$ and $C_{max}$ was constructed. The efficiency of these approaches depends on the number of jobs and the accuracy of the input-output parameters. The core idea of the second approach was to construct a schedule with lower $L_{max}$ value than in the previous step, but use the knowledge, obtained in the previous steps. This allows us to adopt a makespan algorithm to the criterion $L_{max}$ without a substantial increase of the complexity.

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