Metric for the total tardiness minimization problem

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Introduction

Suppose that we have a set $N = \{1, 2, \ldots, n\}$ of n jobs to be processed on a single machine. Preemptions are not allowed. The machine is available since time $t_0 = 0$ and can handle only one job at a time. Job $j \in N$ is available for processing since its *release date* $r_j \geq 0$, its processing requires *processing time* $p_j \geq 0$ time units and should ideally be completed before its *due date* d_j. We will call an instance the set of given parameters: release dates, processing times, and due dates. We will use superscripts to distinguish parameters belonging to different instances. Note that an instance $A = \{r_1^A, \ldots, r_n^A, p_1^A, \ldots, p_n^A, d_1^A, \ldots, d_n^A\}$ can be considered as a vector in $3n$-dimensional space.

Let $S_j(\pi)$ and $C_j(\pi)$ be *the starting and the completion time* of job $j \in N$ in schedule π, respectively. We will consider only *early schedules*, i.e., if $\pi = \{j_1, \ldots, j_n\}$, then $S_{j_1} = \max\{0, r_{j_1}\}$, $S_{j_k} = \max\{r_{j_k}, C_{j_{k-1}}\}, k = 2, 3, \ldots, n$, and $C_j(\pi) = S_j(\pi) + p_j, j \in N$. Thus an early schedule is uniquely determined by a permutation of the jobs of set N. Then let $T_j(\pi) = \max\{0, C_j(\pi) - d_j\}$ be a *tardiness* of job j in schedule π.

The objective is to find an *optimal schedule* $\bar{\pi}$ which minimizes the *total tardiness*, i.e., objective function is $F(\pi) = \sum\limits_{j \in N} T_j(\pi)$. The problem is denoted by $1|r_j|\sum T_j$.

In the paper we propose a new approach for the total tardiness minimization problem. The approach is to construct a polynomially solvable instance B and apply its solution to the given instance A. To evaluate the error of the solution we construct a metric for the considered problem.

For the problem $1|r_j|\sum T_j$ we propose a metric $\rho(A,B)$.

$$\rho(A,B) = n \cdot \max_{j\in N} |r_j^A - r_j^B| + n \cdot \sum_{j\in N} |p_j^A - p_j^B| + \sum_{j\in N} |d_j^A - d_j^B|.$$

This function can be considered as a metric for the problem and bounds difference between optimal values of objective functions of instances A and B.

Metrical approach

Theorem 1. *The function*

$$\rho(A,B) = n \cdot \max_{j\in N} |r_j^A - r_j^B| + n \cdot \sum_{j\in N} |p_j^A - p_j^B| + \sum_{j\in N} |d_j^A - d_j^B|.$$

satisfies the metric axioms.

Theorem 2. *Let $\bar{\pi}^A$ and $\bar{\pi}^B$ be an optimal schedules for instances A and B, respectively. Moreover, let $\tilde{\pi}^B$ be an approximate schedule, subject to*

$$\sum_{j\in N} T_j^B(\tilde{\pi}^B) - \sum_{j\in N} T_j^B(\bar{\pi}^B) \le \delta.$$

Then

$$\sum_{j\in N} T_j^A(\tilde{\pi}^B) - \sum_{j\in N} T_j^A(\bar{\pi}^A) \le 2\rho(A,B) + \delta.$$

The idea of the metrical approach is to find the least distanced in the metric from the given instance A polynomially solvable instance B. Then, by applying known polynomial algorithm to the instance B, one obtains a schedule π^B which can be used as an approximate solution for instance A with error no greater than $2\rho(A,B)$. One can also use approximate solution for the instance B with an absolute error δ as an approximate solution for instance A, in this case the error is not greater that $2\rho(A,B) + \delta$.

Thereby, the problem $1|r_j|\sum T_j$ is reduced to the function $\rho(A,B)$ minimization problem .

Let us search for the instance B in the polynomially solvable class defined by the system of linear inequalities

$$\mathcal{A} \cdot R^B + \mathcal{B} \cdot P^B + \mathcal{C} \cdot D^B \leq H,$$

where $R^B = (r_1^B, \ldots, r_n^B)^T, P^B = (p_1^B, \ldots, p_n^B)^T,\ D^B = (d_1^B, \ldots, d_n^B)^T$, $p_j^B \geq 0, r_j^B \geq 0,\ j \in N,\ ^T$ is transposition symbol, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ – $m \times n$ matrices, and H – a column of m elements.

Then the problem of finding the least distanced from A instance of the given polynomially solvable class can be formulated as follows

$$\text{minimize } f = n \cdot (y^r - x^r) + n \cdot \sum_{j \in N} (y_j^p - x_j^p) + \sum_{j \in N} (y_j^d - x_j^d),$$

subject to

$$\begin{gathered} x^r \leq r_j^A - r_j^B \leq y^r, \\ x_j^p \leq p_j^A - p_j^B \leq y_j^p, \\ x_j^d \leq d_j^A - d_j^B \leq y^d, \\ r_j^B \geq 0, p_j^B \geq 0, j \in N, \\ \mathcal{A} \cdot R^B + \mathcal{B} \cdot P^B + \mathcal{C} \cdot D^B \leq H. \end{gathered}$$

It is the problem of the linear programming, with $7n + 2$ variables: $r_j^B, p_j^B, d_j^B, x_j^p, y_j^p,\ x_j^d, y_j^d, x^r, y^r, j = 1, \ldots, n$.

However, it is not necessary to use algorithms of the linear programming, if there are less complicated ways.

The metrical approach can be applied to other scheduling problems, if a metric function with required properties is constructed.

Lemma 1. *Consider the scheduling problem with following objective function*

$$F(\pi) = \sum_{j \in N} \phi_j(\pi, r_1, \ldots, r_n, p_1, \ldots, p_n, d_j).$$

Then the function

$$\rho(A, B) = \sum_{j \in N} \sum_{i \in N} (R_{ji} |r_j^A - r_j^B| + P_{ji} |p_j^A - p_j^B|) + \sum_{j \in N} D_j |d_j^A - d_j^B|,$$

where $R_{ji} \geq |\frac{\partial \phi_i}{\partial r_j}|, P_{ji} \geq |\frac{\partial \phi_i}{\partial p_j}|, D_j \geq |\frac{\partial \phi_j}{\partial d_j}|$, *can be used as a metric for the problem, and the metrical approach can be applied to find an approximate solution of the problem.*

Computational experiments

We used three polynomially solvable classes in computational experiments. These classes are $\{\mathcal{PR} : p_j = p, r_j = r, j \in N\}$, $\{\mathcal{PD} : p_j = p, d_j = d, j \in N\}$, $\{\mathcal{RD} : r_j = r, d_j = d, j \in N\}$. In the optimal schedules for these classes jobs are processed in the increasing order the free parameter.

Lemma 2. *Minimum of the metric function* $\rho(A, B)$*, where* $B \in \{\mathcal{PR}, \mathcal{PD}, \mathcal{RD}\}$ *can be found in* $O(n)$ *operations.*

To evaluate approximate solutions for both cases we have run computational experiments. For each value of n and each of used polynomially solvable classes 10000 instances were generated. Experiments were performed for $n = 4, 5, \dots, 10$. For each instance, processing times p_j were generated randomly in the interval $[1, 100]$, due dates d_j were generated in the interval $[p_j, \sum_{j \in N} p_j]$, and release dates r_j were generated in the interval $[0, d_j - p_j]$. We used proposed approach to find an approximate solution with value of objective function F_a for each instance, and branch & bound algorithm to find an optimal solution with value of objective function F_o. After we estimated experimental error $\delta = F_a - F_o$ in percentage of the theoretical error, which is doubled value of function $\rho(A, B)$.

All obtained distributions are bell-shaped. Obtained average errors are shown in Table 1. In the $\mathcal{PR}$-case experimental errors averages near 2,5% of the theoretical, in $\mathcal{PD}$-case average error is near 4,5% and in $\mathcal{RD}$-case error grows from 20% to 30% with increasing of n

Conclusion

In the paper we have proposed the new approach to the total tardiness minimization problem. The approach is based on search for the polynomially solvable instance which has a minimal distance in the metric from the original instance. In further research we are going to improve

Table 1: Average experimental error in percentage of the theoretical error

n	$\mathcal{PR}$	$\mathcal{PD}$	$\mathcal{RD}$
4	2,5%	4,6%	20,8%
5	2,6%	4,8%	23,1%
6	2,6%	4,6%	24,6%
7	2,6%	4,7%	26%
8	2,5%	4,6%	27%
9	2,4%	4,7%	27,9%
10	2,4%	4,6%	28,6%

the approach by constructing new metrics and finding new polynomially solvable cases of scheduling problems.

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