Metric for the total tardiness minimization problem

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Introduction

Suppose that we have a set \( N = \{1, 2, \ldots, n\} \) of \( n \) jobs to be processed on a single machine. Preemptions are not allowed. The machine is available since time \( t_0 = 0 \) and can handle only one job at a time. Job \( j \in N \) is available for processing since its release date \( r_j \geq 0 \), its processing requires processing time \( p_j \geq 0 \) time units and should ideally be completed before its due date \( d_j \). We will call an instance the set of given parameters: release dates, processing times, and due dates. We will use superscripts to distinguish parameters belonging to different instances. Note that an instance \( A = \{r_1^A, \ldots, r_n^A, p_1^A, \ldots, p_n^A, d_1^A, \ldots, d_n^A\} \) can be considered as a vector in \( 3n \)-dimensional space.

Let \( S_j(\pi) \) and \( C_j(\pi) \) be the starting and the completion time of job \( j \in N \) in schedule \( \pi \), respectively. We will consider only early schedules, i.e., if \( \pi = \{j_1, \ldots, j_n\} \), then \( S_{j_k} = \max\{0, r_{j_k}\}, \ C_{j_k} = \max\{r_{j_k}, C_{j_{k-1}}\}, k = 2, 3, \ldots, n, \) and \( C_j(\pi) = S_j(\pi) + p_j, j \in N \). Thus an early schedule is uniquely determined by a permutation of the jobs of set \( N \). Then let \( T_j(\pi) = \max\{0, C_j(\pi) - d_j\} \) be a tardiness of job \( j \) in schedule \( \pi \).

The objective is to find an optimal schedule \( \bar{\pi} \) which minimizes the total tardiness, i.e., objective function is \( F(\pi) = \sum_{j \in N} T_j(\pi) \). The problem is denoted by \( 1|r_j|\sum T_j \).

In the paper we propose a new approach for the total tardiness minimization problem. The approach is to construct a polynomially solvable instance \( B \) and apply its solution to the given instance \( A \). To evaluate the error of the solution we construct a metric for the considered problem.
For the problem $1|\text{r}_j|\sum T_j$ we propose a metric $\rho(A, B)$.

$$\rho(A, B) = n \cdot \max_{j \in N} |r_j^A - r_j^B| + n \cdot \sum_{j \in N} |p_j^A - p_j^B| + \sum_{j \in N} |d_j^A - d_j^B|.$$ 

This function can be considered as a metric for the problem and bounds difference between optimal values of objective functions of instances $A$ and $B$.

**Metrical approach**

**Theorem 1.** The function

$$\rho(A, B) = n \cdot \max_{j \in N} |r_j^A - r_j^B| + n \cdot \sum_{j \in N} |p_j^A - p_j^B| + \sum_{j \in N} |d_j^A - d_j^B|.$$ 

satisfies the metric axioms.

**Theorem 2.** Let $\hat{\pi}^A$ and $\hat{\pi}^B$ be an optimal schedules for instances $A$ and $B$, respectively. Moreover, let $\tilde{\pi}^B$ be an approximate schedule, subject to

$$\sum_{j \in N} T_j^B(\tilde{\pi}^B) - \sum_{j \in N} T_j^B(\hat{\pi}^B) \leq \delta.$$ 

Then

$$\sum_{j \in N} T_j^A(\tilde{\pi}^B) - \sum_{j \in N} T_j^A(\hat{\pi}^A) \leq 2\rho(A, B) + \delta.$$ 

The idea of the metrical approach is to find the least distance $d$ in the metric from the given instance $A$ polynomially solvable instance $B$. Then, by applying known polynomial algorithm to the instance $B$, one obtains a schedule $\pi^B$ which can be used as an approximate solution for instance $A$ with error no greater than $2\rho(A, B)$. One can also use approximate solution for the instance $B$ with an absolute error $\delta$ as an approximate solution for instance $A$, in this case the error is not greater that $2\rho(A, B) + \delta$.

Thereby, the problem $1|\text{r}_j|\sum T_j$ is reduced to the function $\rho(A, B)$ minimization problem.
Let us search for the instance $B$ in the polynomially solvable class defined by the system of linear inequalities

$$A \cdot R^B + B \cdot P^B + C \cdot D^B \leq H,$$

where $R^B = (r^B_1, \ldots, r^B_n)^T, P^B = (p^B_1, \ldots, p^B_n)^T, D^B = (d^B_1, \ldots, d^B_n)^T, p^B_j \geq 0, r^B_j \geq 0, j \in N$, $T$ is transposition symbol, $A, B, C - m \times n$ matrices, and $H$ - a column of $m$ elements.

Then the problem of finding the least distanced from $A$ instance of the given polynomially solvable class can be formulated as follows

$$\begin{align*}
\text{minimize } & f = n \cdot (y^r - x^r) + n \cdot \sum_{j \in N} (y^p_j - x^p_j) + \sum_{j \in N} (y^d_j - x^d_j), \\
\text{subject to } & x^r \leq r^A_j - r^B_j \leq y^r, \\
& x^p \leq p^A_j - p^B_j \leq y^p, \\
& x^d \leq d^A_j - d^B_j \leq y^d, \\
& r^B_j \geq 0, p^B_j \geq 0, j \in N, \\
& A \cdot R^B + B \cdot P^B + C \cdot D^B \leq H.
\end{align*}$$

It is the problem of the linear programming, with $7n + 2$ variables: $r^B_j, p^B_j, d^B_j, x^r_j, y^r_j, x^p_j, y^p_j, x^d, y^d, j = 1, \ldots, n$.

However, it is not necessary to use algorithms of the linear programming, if there are less complicated ways.

The metrical approach can be applied to other scheduling problems, if a metric function with required properties is constructed.

**Lemma 1.** Consider the scheduling problem with following objective function

$$F(\pi) = \sum_{j \in N} \phi_j(\pi, r_1, \ldots, r_n, p_1, \ldots, p_n, d_j).$$

Then the function

$$\rho(A, B) = \sum_{j \in N} \sum_{i \in N} (R_{ji} |r^A_j - r^B_j| + P_{ji} |p^A_j - p^B_j|) + \sum_{j \in N} D_j |d^A_j - d^B_j|.$$
where \( R_j \geq |\frac{\partial \phi_i}{\partial r_j}|, P_j \geq |\frac{\partial \phi_i}{\partial p_j}|, D_j \geq |\frac{\partial \phi_i}{\partial d_j}| \), can be used as a metric for the problem, and the metrical approach can be applied to find an approximate solution of the problem.

**Computational experiments**

We used three polynomially solvable classes in computational experiments. These classes are \( \{PR : p_j = p, r_j = r, j \in N\} \), \( \{PD : p_j = p, d_j = d, j \in N\} \), \( \{RD : r_j = r, d_j = d, j \in N\} \). In the optimal schedules for these classes jobs are processed in the increasing order the free parameter.

**Lemma 2.** Minimum of the metric function \( \rho(A, B) \), where \( B \in \{PR, PD, RD\} \) can be found in \( O(n) \) operations.

To evaluate approximate solutions for both cases we have run computational experiments. For each value of \( n \) and each of used polynomially solvable classes 10000 instances were generated. Experiments were performed for \( n = 4, 5, \ldots, 10 \). For each instance, processing times \( p_j \) were generated randomly in the interval \([1, 100]\), due dates \( d_j \) were generated in the interval \([p_j, \sum_{j \in N} p_j]\), and release dates \( r_j \) were generated in the interval \([0, d_j - p_j]\). We used proposed approach to find an approximate solution with value of objective function \( F_a \) for each instance, and branch & bound algorithm to find an optimal solution with value of objective function \( F_o \). After we estimated experimental error \( \delta = F_a - F_o \) in percentage of the theoretical error, which is doubled value of function \( \rho(A, B) \).

All obtained distributions are bell-shaped. Obtained average errors are shown in Table 1. In the \( PR \)-case experimental errors averages near 2,5% of the theoretical, in \( PD \)-case average error is near 4,5% and in \( RD \)-case error grows from 20% to 30% with increasing of \( n \).

**Conclusion**

In the paper we have proposed the new approach to the total tardiness minimization problem. The approach is based on search for the polynomially solvable instance which has a minimal distance in the metric from the original instance. In further research we are going to improve
Table 1: Average experimental error in percentage of the theoretical error

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<th>RD</th>
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<tr>
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<td>4.6%</td>
<td>28.6%</td>
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the approach by constructing new metrics and finding new polynomially solvable cases of scheduling problems.

The authors were supported by the Russian Foundation for Basic Research (projects no. 11-08-01321 and 11-08-13121).

REFERENCES