Metric and Approximated Solution of the Single Machine Total Tardiness Minimization Scheduling Problem

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Abstract: In this paper, we consider the NP-hard minimizing total tardiness on a single machine scheduling problem. We propose a metric for that problem and present a polynomial approximation scheme based on search for the polynomially solvable instance which has a minimal distance from an initial instance.

Keywords: Scheduling algorithm, total tardiness, metric, approximation scheme.

1. INTRODUCTION

In the paper, we consider the following problem. Suppose we have a set $N = \{1, 2, \ldots, n\}$ of n jobs to be processed on a single machine. Preemptions are not allowed. The machine is available since time $t_0 = 0$ and can handle only one job at a time. Job $j \in N$ is available for processing since its *release date* $r_j \geq 0$, its processing requires *processing time* $p_j \geq 0$ units and should ideally be completed before its *due date* d_j. We will call an instance the set of given parameters: release dates, processing times, and due dates. We will use superscripts to distinguish parameters belonging to different instances. Note that an instance $A = \{r_1^A, \ldots, r_n^A, p_1^A, \ldots, p_n^A, d_1^A, \ldots, d_n^A\}$ can be considered as vector in $3n$-dimensional space.

Let $S_j(\pi)$ and $C_j(\pi)$ be *the starting and the completion time* of job $j \in N$ in schedule π, respectively. Further we will omit the argument in brackets whenever it is clear from context. We will consider only *early schedules* (sequences), i.e., if $\pi = (j_1, \ldots, j_n)$, then

$$S_{j_1} = \max\{0, r_{j_1}\},$$
$$S_{j_k} = \max\{r_{j_k}, C_{j_{k-1}}\}, k = 2, 3, \ldots, n,$$

and

$$C_j(\pi) = S_j(\pi) + p_j, j \in N.$$

Thus an early schedule is uniquely determined by a permutation of the jobs of set N. Then let $T_j(\pi) = \max\{0, C_j(\pi) - d_j\}$ be a *tardiness* of job j in schedule π.

The objective is to find an *optimal schedule* $\bar{\pi}$ which minimizes the *total tardiness*, i.e., objective function is $F(\pi) = \sum_{j\in N} T_j(\pi)$. Graham *et. al.* (1979) denoted this problem by $1|r_j|\sum T_j$.

Du and Leung (1990) showed that the special case $1||\sum T_j$ of the problem is NP-hard in ordinary sense. The pseudopolynomial algorithm of $O(n^4 \sum p_j)$ operations for that case was found by Lawler (1977), later Lawler (1982) constructed fully polynomial approximation scheme of $O(\frac{n^7}{\epsilon})$ operations for the same case. For the case $1||\sum T_j$ with additional restrictions:

$$p_1 \geq p_2 \geq \cdots \geq p_n,$$
$$d_1 \leq d_2 \leq \cdots \leq d_n,$$

Lazarev and Werner (2009) proposed a pseudopolynomial algorithm of $O(n^2 \sum p_j)$ operations. Baptiste (2000) proposed polynomial algorithm of $O(n^7)$ operations for $1|r_j; p_j = p|\sum T_j$ problem.

In the paper we propose an approximation scheme for the total tardiness minimization problem. In the scheme we construct a polynomially solvable instance B and apply its solution to the given instance A. To evaluate the error of the solution we construct a metric for the considered problem. For the $1||\sum T_j$ problem the metric was constructed by Lazarev and Kvaratskheliya (2010). For the problem $1|r_j|\sum T_j$ we propose metric $\rho(A, B)$

$$\rho(A,B) = n\cdot\max_{j\in N}|r_j^A - r_j^B| + n\cdot\sum_{j\in N}|p_j^A - p_j^B| + \sum_{j\in N}|d_j^A - d_j^B|.$$

2. METRIC FOR THE SPACE OF INSTANCES

Further we will use the following inequality

Proposition 1. For any real numbers a, b, c, d we have

$$|\max\{a,b\} - \max\{c,d\}| \leq \max\{|a-c|, |b-d|\}. \quad (1)$$

Corollary 1. Let $\sum_{j\in N} T_j^A$ and $\sum_{j\in N} T_j^B$ be total tardinesses for instances A and B under a sequence π, respectively. By applying (1) to their difference, we obtain

$$|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B| \leq \sum_{j\in N} |(C_j^A - d_j^A) - (C_j^B - d_j^B)|. \quad (2)$$

Corollary 2. Let A and B be instances with $p_j^A = p_j^B$ for all $j \in N$ and $\pi = (j_1, j_2, \dots, j_n)$ be an applied schedule. According to (1), we obtain

$$|C_{j_1}^A - C_{j_1}^B| \le |r_{j_1}^A - r_{j_1}^B|,$$
$$|C_{j_k}^A - C_{j_k}^B| \le \max\{|r_{j_k}^A - r_{j_k}^B|, |C_{j_{k-1}}^A - C_{j_{k-1}}^B|\},\ k \ge 2,$$

i.e., for all $j \in N$ we have

$$|C_j^A - C_j^B| \le \max_{i \in N} |r_i^A - r_i^B|. \quad (3)$$

Corollary 3. Let A and B be instances with $r_j^A = r_j^B$ for all $j \in N$ and $\pi = (j_1, j_2, \dots, j_n)$ be an applied schedule. According to (1), we obtain

$$|C_{j_k}^A - C_{j_k}^B| \le |C_{j_{k-1}}^A - C_{j_{k-1}}| + |p_{j_k}^A - p_{j_k}^B|, \text{ for } k \ge 2,$$
$$|C_{j_1}^A - C_{j_1}^B| = |p_{j_1}^A - p_{j_1}^B|,$$

then, using mathematical induction, we have

$$|C_j^A - C_j^B| \le \sum_{i \in N} |p_i^A - p_i^B|. \quad (4)$$

for all $j \in N$.

Lemma 1. Let A and B be instances with the same processing times and due dates

$$p_j^A = p_j^B,$$
$$d_j^A = d_j^B,$$

for all $j \in N$. Then for any schedule π we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le n \cdot \max_{j \in N} |r_j^A - r_j^B|. \quad (5)$$

Proof.

Using (2), we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le \sum_{j \in N} |C_j^A - C_j^B|,$$

then by using (3), we obtain

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le n \cdot \max_{j \in N} |r_j^A - r_j^B|.$$

Lemma 2. Let A and B be instances with the same release times and due dates

$$r_j^A = r_j^B,$$
$$d_j^A = d_j^B,$$

for all $j \in N$. Then for any schedule π we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le n \cdot \sum_{j \in N} |p_j^A - p_j^B|. \quad (6)$$

Proof.

Using (2), we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le \sum_{j \in N} |C_j^A - C_j^B|,$$

then by using (4), we obtain

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le n \cdot \sum_{j \in N} |p_j^A - p_j^B|.$$

Lemma 3. Let A and B be instances with the same release dates and processing times

$$r_j^A = r_j^B,$$
$$p_j^A = p_j^B,$$

for all $j \in N$. Then for any schedule π we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le \sum_{j \in N} |d_j^A - d_j^B|. \quad (7)$$

Proof.

Using (2), we have

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le \sum_{j \in N} |d_j^A - d_j^B|.$$

Lemma 4. The function defined on the $3n$-dimensional space of instances

$$\rho(A, B) = n \cdot \max_{j \in N} |r_j^A - r_j^B| + n \cdot \sum_{j \in N} |p_j^A - p_j^B| + \sum_{j \in N} |d_j^A - d_j^B| \quad (8)$$

satisfies the metric axioms.

Proof.

It is obvious that for any A, B we have $\rho(A, B) \ge 0$ and $\rho(A, B) = 0$, iff $A = B$, and $\rho(A, B) = \rho(B, A)$. Let us check the triangle inequality for $\rho(A, B)$.

For any $r_j^A, r_j^B, r_j^C, j \in N$.

$$|r_j^A - r_j^B| \le |r_j^A - r_j^C| + |r_j^B - r_j^C|,$$

then

$$\max_{j \in N} |r_j^A - r_j^B| \le \max_{j \in N} |r_j^A - r_j^C| + \max_{j \in N} |r_j^B - r_j^C|.$$

Then for any instances A, B, C we have

$$n \cdot \max_{j \in N} |r_j^A - r_j^B| \le n \cdot \max_{j \in N} |r_j^A - r_j^C| + n \cdot \max_{j \in N} |r_j^B - r_j^C|,$$
$$n \cdot \sum_{j \in N} |p_j^A - p_j^B| \le n \cdot \sum_{j \in N} |p_j^A - p_j^C| + n \cdot \sum_{j \in N} |p_j^B - p_j^C|,$$
$$\sum_{j \in N} |d_j^A - d_j^B| \le \sum_{j \in N} |d_j^A - d_j^C| + \sum_{j \in N} |d_j^B - d_j^C|,$$

and

$$\rho(A, B) \le \rho(A, C) + \rho(B, C).$$

According to Lemma 4, function $\rho(A, B)$ can be considered as a distance between instances A and B.

3. APPROXIMATION SCHEME

Lemma 5. For any instances A, B and a schedule π

$$|\sum_{j \in N} T_j^A - \sum_{j \in N} T_j^B| \le \rho(A, B). \quad (9)$$

Proof.

Let C^d be an instance which has the same processing times and release dates with the instance A and the same due dates with B. Moreover, let C^r be an instance which has the same release dates with the instance A and the same processing times and due dates with B. Then using (5)-(7) we obtain

$$|\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^B| \le |\sum_{j\in N} T_j^A - \sum_{j\in N} T_j^{C^d}| + |\sum_{j\in N} T_j^{C^d} - \sum_{j\in N} T_j^{C^r}| + |\sum_{j\in N} T_j^{C^r} - \sum_{j\in N} T_j^B| \le \sum_{j\in N} |d_j^A - d_j^B| + n\cdot\sum_{j\in N}|p_j^A - p_j^B| + n\cdot\max_{j\in N}|r_j^A - r_j^B| = \rho(A,B).$$

Theorem 1. Let $\bar{\pi}^A$ and $\bar{\pi}^B$ be optimal schedules for instances A and B, respectively. Moreover, let $\tilde{\pi}^B$ be an approximate schedule for B

$$\sum_{j\in N} T_j^B(\tilde{\pi}^B) - \sum_{j\in N} T_j^B(\bar{\pi}^B) \le \delta. \quad (10)$$

Then

$$\sum_{j\in N} T_j^A(\tilde{\pi}^B) - \sum_{j\in N} T_j^A(\bar{\pi}^A) \le \delta + 2\rho(A,B). \quad (11)$$

Proof.

From (9), (10) we have

$$\sum_{j\in N} T_j^A(\tilde{\pi}^B) - \sum_{j\in N} T_j^A(\bar{\pi}^A) \le \sum_{j\in N} T_j^B(\tilde{\pi}^B) - \sum_{j\in N} T_j^B(\bar{\pi}^A) + 2\rho(A,B) \le 2\rho(A,B) + \delta + \sum_{j\in N} T_j^B(\bar{\pi}^B) - \sum_{j\in N} T_j^B(\bar{\pi}^A) \le 2\rho(A,B) + \delta.$$

Theorem 1 allows one to find a polynomial approximate solution for $1|r_j|\sum T_j$ according to the following scheme.

The idea of the approximation scheme is to find the least distanced from the given instance A polynomially solvable instance B. Then, by applying a known polynomial algorithm to the instance B, one obtains a schedule $\bar{\pi}^B$ which can be used as an approximate solution for instance A with error not more than $2\rho(A,B)$. One can also use approximate solution for the instance B with an absolute error δ as an approximate solution for instance A, in this case the error is not more than $2\rho(A,B) + \delta$.

Thereby, the problem $1|r_j|\sum T_j$ is reduced to the function $\rho(A,B)$ minimization problem.

Let us search for the instance B in the polynomially solvable class defined by the system of linear inequalities

$$\mathcal{A}\cdot R^B + \mathcal{B}\cdot P^B + \mathcal{C}\cdot D^B \le H,$$

where $R^B = (r_1^B,\dots,r_n^B)^T, P^B = (p_1^B,\dots,p_n^B)^T$, $D^B = (d_1^B,\dots,d_n^B)^T$, $p_j^B \ge 0, r_j^B \ge 0, j\in N$, ^{T} is transposition symbol, $\mathcal{A},\mathcal{B},\mathcal{C}$ – $m\times n$ matrices, and H – a column of m elements.

Then the problem of finding the least distanced from A instance of the given polynomially solvable class can be formulated as follows

$$\text{minimize } n\cdot(y^r - x^r) + n\cdot\sum_{j\in N}(y_j^p - x_j^p) + \sum_{j\in N}(y_j^d - x_j^d), \quad (12)$$

subject to

$$x^r \le r_j^A - r_j^B \le y^r,$$
$$x_j^p \le p_j^A - p_j^B \le y_j^p,$$
$$x_j^d \le d_j^A - d_j^B \le y^d,$$
$$r_j^B \ge 0, p_j^B \ge 0, j\in N,$$
$$\mathcal{A}\cdot R^B + \mathcal{B}\cdot P^B + \mathcal{C}\cdot D^B \le H.$$

It is the problem of the linear programming, with $7n+2$ variables: $r_j^B, p_j^B, d_j^B, x_j^p, y_j^p, x_j^d, y_j^d, x^r, y^r, j = 1,\dots,n$.

However, it is not necessary to use algorithms of the linear programming, if there are less complicated ways.

4. EXAMPLES OF THE APPROXIMATION SCHEME

Let $\mathcal{PR}$ denote the class of instances with $p_j = p, r_j = r, j\in N$, $\mathcal{PD}$ denote the class with $p_j = p, d_j = d, j\in N$, and $\mathcal{RD}$ denote the class with $r_j = r, d_j = d, j\in N$. Those classes are polynomially solvable. In the optimal schedules jobs are processed in the increasing order of their due dates for $\mathcal{PD}$ class, in the increasing order of their release dates for $\mathcal{PR}$, and in the increasing order of their processing times for $\mathcal{RD}$.

If in the approximation scheme one searches for the instance B in class $\mathcal{PR}$, one has to minimize the function

$$f(p,r) = n\cdot\sum_{j\in N}|p_j^A - p| + n\cdot\max_{j\in N}|r_j^A - r|. \quad (13)$$

Lemma 6. Function (13) has a minimum at point ($p\in\{p_1^A,\dots,p_n^A\}, r = \frac{r_{max}^A + r_{min}^A}{2}$), where $r_{max}^A = \max_{j\in N} r_j^A$, $r_{min}^A = \min_{j\in N} r_j^A$.

Proof. Function $f(p,r)$ can be divided into two functions of one variable: $f(r,p) = f(r) + f(p)$, where

$$f(r) = n\cdot\max_{j\in N}|r_j^A - r|,$$
$$f(p) = n\cdot\sum_{j\in N}|p_j^A - p|.$$

We first consider the function $f(r)$. Note that $f(r)$ can be rewritten in the equivalent form

$$f(r) = n\cdot\max_{j\in N}|r_j^A - r| = n\cdot\max\{r - r_{min}^A, r_{max}^A - r\} = n\cdot\frac{r_{max}^A - r_{min}^A}{2} + n\cdot|r - \frac{r_{max}^A + r_{min}^A}{2}|.$$

It is obvious, that $f(r)$ has the minimum at $r = \frac{r_{max}^A + r_{min}^A}{2}$.

Now we prove that function $f(p)$ has the minimum at $p\in\{p_1^A,\dots,p_n^A\}$. The function $f(p)$ may have a minimum at the points where function is not differentiable, i.e. at the points $p_1^A, p_p^A, \ldots, p_n^A$. Then the lemma holds. If the function has a minimum at the point where derivative is zero, then, because function $F(p^B)$ is linear on the interval $(p_j^A, p_{j+1}^A), j = 0, \ldots, n+1,\ p_0 = -\infty,\ p_{n+1} = \infty$, the function is a constant on that interval and values p_j^A and p_{j+1}^A are points of minimum too.

If in the approximation scheme one searches for the instance B in class $\mathcal{PD}$, one has to minimize the function

$$g(p, d) = n \cdot \sum_{j \in N} |p_j^A - p| + \sum_{j \in N} |d_j^A - d|.$$

Lemma 7. Function $g(p, d)$ has a minimum at point $(p \in \{p_1^A, \ldots, p_n^A\}, d \in \{d_1^A, \ldots, d_n^A\})$.

Proof. The lemma can be proved analogously to Lemma 6.

And if in the approximation scheme one searches for the instance B in class $\mathcal{RD}$, one has to minimize the function

$$h(r, d) = n \cdot \max_{j \in N} |r_j^A - r| + \sum_{j \in N} |d_j^A - d|.$$

Lemma 8. Function $h(r, d)$ has a minimum at point $(r = \frac{r_{max}^A + r_{min}^A}{2}, d \in \{d_1^A, \ldots, d_n^A\})$.

Proof. The lemma can be proved analogously to Lemma 6.

Therefore, minimums of $f(r, p)$, $g(p, d)$ and $h(r, d)$ can be found in $O(n)$ operations.

So, we have three variants of the proposed scheme: with the use of $\mathcal{PR}$, $\mathcal{PD}$ and $\mathcal{RD}$ classes. To evaluate approximate solutions for both cases we have run computational experiments. 10000 instances were generated for each value of n. Experiments were performed for $n = 4, 5, \ldots, 10$. For each instance, processing times p_j were generated randomly in the interval $[1, 100]$, due dates d_j were generated in the interval $[-100, 100]$, and release dates r_j were generated in the interval $[0, 100]$. We used proposed scheme to find an approximate solution with value of objective function F_a for each instance, and branch & bound algorithm to find an optimal solution with value of objective function F^*. After we estimated experimental error $\Delta = F_a - F^*$ in percentage of the theoretical error, which is doubled value of function $f(r, p)$ or $g(p, d)$ for cases with $\mathcal{PR}$ and $\mathcal{PD}$ classes, respectively.

All obtained distributions are bell-shaped. The typical distribution of experimental error is shown in Fig. 1. In both cases distributions narrow with increasing of n. In the $\mathcal{PR}$-case experimental errors averages near 19% of the theoretical. In the $\mathcal{RD}$-case experimental errors grows from 15% to 23% of the theoretical. The least ratio of the experimental to the theoretical error has been obtained in $\mathcal{PD}$-case — it does not exceed 30% of theoretical, though its average grows from 5% to 10% with increasing of n. Obtained average errors are shown in Table 1.

5. CONCLUSION

In the paper we have proposed the new approximation scheme for the total tardiness minimization problem. The scheme is based on search for the polynomially solvable instance which has a minimal distance in the metric from the original instance.

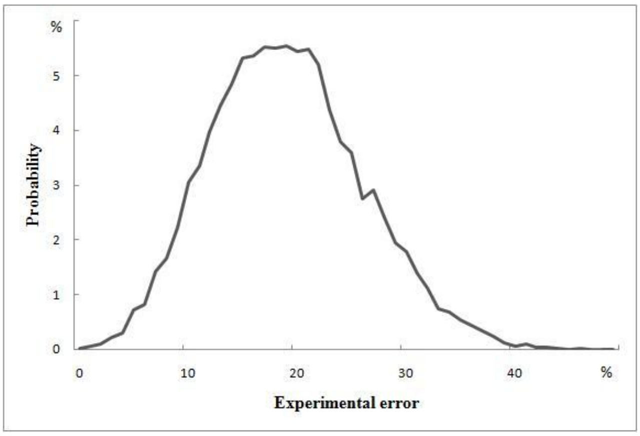

Fig. 1. Distribution of experimental error in percentage of the theoretical error

Table 1. Average experimental error in percentage of the theoretical error

n	$\mathcal{PR}$-case	$\mathcal{PD}$-case	$\mathcal{RD}$-case
4	19%	4,5%	15%
5	19,5%	6,2%	17,2%
6	19,2%	7,3%	18,4%
7	19,6%	8,5%	19,4%
8	19,3%	9,2%	20,7%
9	19,4%	10%	21,7%
10	19%	10,5%	22,5%

In further research the scheme can be applied to other scheduling problems. One can also improve the scheme by constructing new metrics and finding new polynomially solvable cases of scheduling problems.

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