Linearized Model of Volatile Equilibrium in Real Economics

Oleg I. Krivosheev
Laboratory of Intellectual Control Systems and Modelling
V.A. Trapeznikov Institute of Control Sciences of RAS
Moscow, Russia
o-krivosheev@yandex.ru

***Abstract*—We develop a theory connecting leverage and volatility in almost the same way as private goods volumes are in dual interconnection with their prices. But the corresponding (substantially dynamical non-linear) theory is technically too complex both for operating and for results understanding, so here we develop a substantially simplified bi-linear isomorphic to general volatile equilibrium model but using only matrixes and 3-dimensional tensors instead of functional spaces and multidimensional phase diagrams. Regardless of the difficulties expected in the general case, this analytical system is appeared to be exactly solvable under some acceptable conditions.**

Keywords*—*general equilibrium, volatility, leverage, finance

I. Introduction

Micro- and macroeconomics are distinguished not by the scale of the systems they try to consider, but because Arrow-McKenzie-Debreu's main microeconomic model is a natural exchange model, with no finance at all and no real money (although with prices). And to regard finance and money AD-AS -IS-LM models are introduced and the overall mixture of the two heterogeneous parts is called synthesis - in our case, the neoclassical one. In this work, we not only conceptually solve the problem of unique theory absence (following [1]), but we also make (under certain assumptions) this solution analytically solvable. The problem we talk about did appear 150 years ago, when Walras formulated his famous general equilibrium model, and 80 years later (in the 1950s) this model was "upgraded" by Arrow and colleges to the contemporary form. Later, J. Gianakopulos in the late 1990s introduced a collateral equilibrium, that permitted a description and calculation of an optimal share of own capital or, hence, the financial leverage, (that is inverse to one). Still, we cannot completely be admitted with the collateral equilibrium model as it has no explanation for the volatility rate, that determines the leverage, making the mathematical system under-defined or not defined.

II. Model of Volatile Equilibrium

In our approach, there exist two individual optimization problems that are also collective constraint redistribution problems on a whole economy scale. The 1st is a redistribution of stability constraint (common) resource we here discuss and the 2nd is a well-known typical private good redistribution being solved in the traditional general equilibrium - GE model. Mostly we are interested in the first problem, but to start with we make several remarks on the second one. Farther, we will consider general equilibrium in two senses 1) traditional GE with inter-temporal choice, when amounts of capital and investment solutions are defined. Though in this case, we have a natural exchange model. 2) We may assume each time static general equilibrium with no internal investment solutions, that are defined by the higher-level model, that takes into account dynamics of the economic system and corresponding risks, both depending on collective investment (and hence financial) strategies. Here we come to the stability constraint, which leads to a problem of stability reserve redistribution in a condition where this reserve is a common good redistribution with volatility as a quasi-price of it.

So, we start from a general equilibrium solution as from the first step approximation. Then we take into account that every set of collective financial (and hence investment) strategies lead to some rate of volatility, that is (taking into account a real asset's expected return rate) immediately permits us to calculate the best response of any economic agent in terms of its investment (financial) strategy, that permits finally mathematically define a model using the Nash equilibrium notion or stationary point of investment strategies to itself through the volatility as an intermediate variable.

As for investment strategies, we consider one as an object of some low-parametrical set. In an elementary case, there considered only two simple parameters: a share of assets in riskless instruments (such as cash or treasury bonds) - let denote it by c, and the rest share $1-c$ is invested at some leverage l (usually more than 1), that permits to get the extended return on own capital, but at the cost of possibly non-zero losses due to bankruptcy flow of some intensity that is either taken into account in bank rate surplus an argent permanently pays on borrowed capital and also from time to time bankruptcy events lead to the own capital falls to the amount of bankruptcy protected riskless reserves, that is an as well non-willing situation to the investor, he wants to be minimized

Let's mathematically describe this strategy.

If we denote

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l - leverage,

c - riskless reserve share

i - mean return on physical capital

i_c - mean return on own capital

$\lambda(l)$ - the intensity of bankruptcy

than

$i_c = (1-c)l(i-\lambda(l)) + \lambda(l)\ln c$, where obviously $\ln c < 0$ $(0 \le c \le 1)$.

Assuming $\lambda(0)=0$ (and even $\lambda(1)=0$) while, if l is sufficiently high $i < \lambda(l)$ we get a maximization problem

$$i_c = (1-c)l(i-\lambda(l)) + \lambda(l)\ln c \to \max_{l\ge 1,\quad 1\ge c\ge 0} \quad (1)$$

for a single- or malty-peak function, where the central term after i is the intensity of bankruptcy $\lambda(l)$, that is calculated using the formula

$$\lambda(\hat{l}, \rho_\lambda(T,\Delta i)) = \int_{l_{MAX}(T,\Delta i)<l} \rho_\lambda(T,\Delta i)dTd\Delta i \, , (2)$$

where $\rho_\lambda(T,\Delta i)$ - is a density function of Poisson's (malty)flow discrete or continuous intensity distribution of shocks with parameters T - time length and Δi - internal rate of return of physical capital under bank rate drop-down and the bankruptcy event happens when

$$\hat{l} > \hat{l}_{MAX}(T,\Delta i) = \frac{d}{\Delta i(1-e^{-d\Delta T})} \, , (2')$$

where d is a depreciation rate. The last formula arises from considerations illustrated in Fig.1

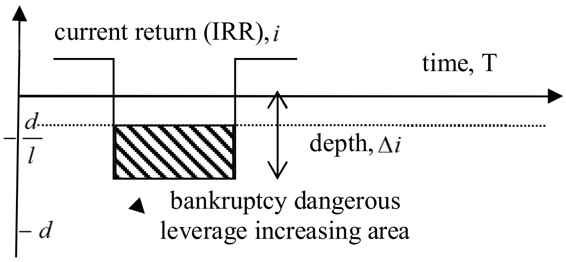

Fig. 1. For each technology every crisis we approximate by the rectangle is characterized by time length and depth.

If $i(t)l > -d$ the investor can shrink investment so he can keep leverage constant. But if $i(t)l$ is too low leverage may rise to infinity, which would mean that debt exceeds physical assets and firm value is negative or not positive.

In the pre-default state i.e., when $i(t)l < -d$ physical capital K shrinks at the maximal rate $-d$

$\frac{d}{dt}K = -dK$ while debt D decreases slower

$\frac{d}{dt}D = -p_K(d+i(t))K$ where p_K - is a physical capital price that leads to this formula if we ask $D(T) - p_K K(T) = 0$, while $\hat{l}_{MAX}(T,\Delta i) = \frac{K(0)p_K}{K(0)p_K - D(0)}$.

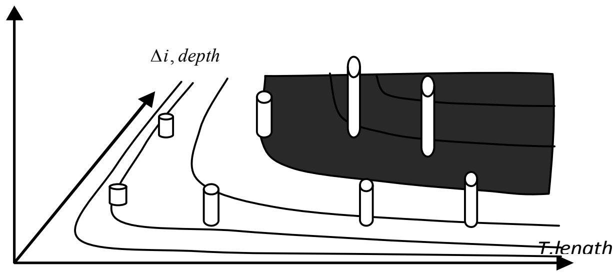

Fig. 2. Crisis event Poisson flow density $\rho_\lambda(T,\Delta i)$ with an area marked in dark representing $\lambda(\hat{l}) = \int_{l_{MAX}(T,\Delta i)<l} \rho_\lambda(T,\Delta i)dTd\Delta i$ integration region.

The $\rho_\lambda(T,\Delta i)$ function is calculated from the deterministic (maybe stochastic) differential multidimensional system

$$\begin{cases} \beta \frac{d}{dt}\vec{p} = \Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K}) \\ \frac{d}{dt}\vec{K} = -[\vec{d}]\vec{K} + \vec{Q}_I^{\,C}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K}) \\ \frac{d}{dt}\vec{D} = \vec{I}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K}) - \vec{\Pi}(\vec{p},\vec{K}) - [\vec{\lambda}]\vec{D} \end{cases} , (3)$$

where

$\Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K})$ - is a supply-demand difference

$\vec{p}$ - price vector,

$\vec{\vartheta}$ - (long-term) market growth rates,

$\vec{K}$ - physical capital (production capacities),

d - depreciation rate,

Q_I^C - physical investment,

$\vec{D}$ - debts in each technology,

λ - is either acts as a risky debt percentage rate b assuming the bank's stockholder's capital i_c and bank riskless return rate is 0 for simplicity and thus $b=\lambda$ (that is not the case with in real situation and we should properly write $b = \lambda + \frac{i_C}{(1-c_{bank})l_{bank}}$, where l_{bank} and c_{bank} are corresponding parameters of banker strategies that are strictly speaking diverse to different borrowers). So, in the assumption $b=\lambda$

$D\lambda$ - is a debt price in the case of well-enough informed agents

$\vec{I}$ - nominal investment,

$\vec{\Pi}$ - income without current expenses that can be re-invested or spent to reduce leverage by partial debt payment,

this Π is either in very close interconnection with the physical return rate i we permanently use

$$i = \frac{\Pi}{p_K K} - d \text{ or}$$

$$i = \frac{\pi}{p_K} - d \text{, where } \pi = \frac{\Pi}{K}$$

p_K - physical capital price

and the most valuable for the consequent is $\vec{l}_{12}$:

$\vec{l}_{12} = (\hat{\vec{l}}, \vec{l})$ is a bi-vector of a goal $\hat{\vec{l}}$ and actual $\vec{l}$ leverage. Here the goal leverage $\hat{\vec{l}}$ acts as a parameter and $\vec{l}$ is still a variable (depending on $\vec{D}$ and $\vec{K}$ by $l = \frac{Kp_K}{Kp_K - D}$).

Generally, we assume that the investor reserves a c-share of own capital in a riskless asset, that is not considered an investor's collateral in the case of bankruptcy. Then the leverage is $l_{eff} = (1-c)l$ (or $\vec{l}_{eff} = (E - [\vec{c}])\vec{l} = \vec{l} - [\vec{c}]\vec{l}$, where $[\vec{c}]$ is a diagonal matrix using $\vec{c}$ - vector as a diagonal).

Thus, we define an individual "demand" on leverage

$l_{eff} = (1-c)l$ depending not on price but volatility and corresponding integral demand, using $(\vec{l}_{eff}, \vec{K})$ pair; in the same way, we define a reversed supply function as volatility as a function of $(\vec{l}_{eff}, \vec{K})$ pair according to ODE (3).

$\rho_\lambda(T, \Delta i)(K, i, \hat{l}_{eff})$, where mean $\vec{K}$ and expected $\vec{i}$ are determined from the initial arrow model, taking into account i_c - the own capital return rate depending on $\vec{\vartheta}$ by

$i_c = \langle \vec{\vartheta} \rangle$ - as a mean growth rate. According to it mean i

$i = \frac{\Pi}{p_K K} - d$ for each technology should be at the level that i_c according to (1) exactly achieves the universal $i_c = \langle \vec{\vartheta} \rangle$. This produces a restriction on the (mean) $\vec{K}$ level that is a regulator of the physical capital mean return rate i.

The main aspect of the model is a supply part $\rho_\lambda(T, \Delta i)(K, i, \hat{l}_{eff})$, that according to that could be written as $\rho_\lambda(T, \Delta i)(\hat{\vec{l}})$ since $\vec{K}$ and i are either dependent on $\lambda(\rho_\lambda(T, \Delta i))$ and on long-term own capital return rate $i_c = \langle \vec{\vartheta} \rangle$.

Here we omitted two technical difficulties (that can be passed at fast reading) -

First is how one should calculate p_K - physical capital price. To make our calculus strictly correct, it should be smoothed over the period of physical capital creation with a product of weights first taking into account the amount of investment at each time and second obviously taking into account total depreciation that had accumulated from that time. This too-complex method may be in several steps simplified, and the simplified variant is to take into account only long-term GE physical capital prices p_K^{Eq}.

And the second problem is how in an economically reasonable way investments should be redistributed if to achieve goal mean i (or $\langle i \rangle$ where brackets $\langle \ \rangle$ denote mean), one should invest less or more than he gets at current technology. Inter-technological and inter-branch capital reflows should solve this problem and should hold i_c at the same rate. We suggest the simplest solution for the case of single mode economic cycle: every branch (and every technology) in the long perspective should approximately invest at its native market growth rate (holding corresponding $\langle i \rangle$), and the rest investment (if excessive) should be delivered among faster-growing markets (and technologies), and in the other case necessary additional investments should be in the same way taken from slowly growing markets. So, we obtain one more diagonal matrix $F = [\vec{f}]$ with elements in simple inaccurate explanation equal to the ratio of each market growth rate to the economy mean growth rate, but in a close-to-reality formula, one should make a vital remark that we assume each growth rate is counted from negative depreciation rate starting point: so, we have $\vartheta + d$ values that are redistributed according to the

$$f = \frac{\vartheta + d}{\langle \vartheta + d \rangle}$$

and F is a corresponding diagonal matrix.

So, $\vec{l}_{eff} = [\vec{f}][1 - \vec{c}]\vec{l}$

and in the scalar form, we write

$\hat{l}_{eff} = f(1-c)\hat{l}$.

In the case of equal growth rates, we have $f = 1$. Further, for the goal of simplification, we consider only this case.

III. Problems Of the General Model

The main difficulty is that to calculate $\rho_\lambda(T, \Delta i)$ at each $\hat{\vec{l}}$ one should have an exact model of the economic system (digital copy of it) push on this model trajectory calculus for the time containing enough events of crisis with their depth and time-length $(T, \Delta i)$ to build an appropriate approximation of $\rho_\lambda(T, \Delta i)$. In a complex case, it can take not less than ten but rather more than a hundred events. If we consider that events generally happen at a time approximately equal to the mean firm lifetime - about ten years and that this interval is long enough. So, for the tatonnement process, the main difficulty is time. For calculation, the difficulty is that we should build an enormously complex model. To avoid these difficulties in this paper was developed a cascade of simplifications including

1) Turn to a static (algebraic) model

2) Turn to a bi-linear model (or linear one by simultaneously leveraging lengths and price deviations).

IV. Short Insertion On Common Resource (CR) Quasi-Markets

There are four types of goods in the classification of access and competitiveness: private, common, club, and public. The first ones are distributed and produced by standard possibly competitive price mechanisms, but the others are not. Such goods that are competitive but still at (unlimited)common access one call common recourses. If the demand exceeds the amount of such resource like in the case of road situation (or any other mass access system) when the number of requests is higher than the installed capacity, then the quality (the attractiveness) of the common resource falls until the supply and demand become equal. So, the time waist in the queue like a traffic jam is the factor acting as a quasi-price. Exactly that way may act any pollution, like thermal pollution in the overcrowded auditorium or any other density or overload effect. There are more than 5000 cases of common recourses discussed in papers at the present moment. Exactly that type of recourse we recognize when talking about the stability of the price system of the economic system described by (1-3).

V. Non-Linear Algebraic Model

A semi-conservative approach we develop in this section is based on a partial simplification of the supply part.

We formalize the border where the ODE (3) point equilibrium loses stability.

This happens when the price gradient $\frac{\partial}{\partial \vec{p}}$ of the vector of supply-demand difference $\Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K})$ which is the right side of the 1st fastest equation in (3) admits

$Max\,\mathrm{Re}\,Spec\left(\frac{\partial}{\partial \vec{p}}\Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K})\right)$ where the composite 3-step function $Max\,\mathrm{Re}\,Spec$ is a maximal real part of all eigenvalues in the matrix $\left(\frac{\partial}{\partial \vec{p}}\Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K})\right)$

As the 1st approximation, we calculate it at the point of long-term general equilibrium, it depends only on the goal leverage and physical vector in the subsequent for

$$C=\frac{\partial}{\partial \vec{p}}\Delta\vec{Q}(\vec{p},\vec{\vartheta},\vec{l}_{12},\vec{K})=C_{00}+\sum_k C_{0k}K_k+\sum_k C_{1k}K_k l_k$$

where C_{00}, C_{0k}, C_{1k} - matrixes, k - a technological number (index).

The growth of the matrix term $\sum_k C_{1k}K_k l_k$ here is responsible for oscillations born (we will also call this term integral leverage without a definition of the way how it can be made a scalar).

We may also phenomenologically assume that volatility depends on the oldest eigenvalue $\Lambda_{R\max}=Max\,\mathrm{Re}\,SpecC(\hat{l})$ so, one could write the volatility

$\rho_\lambda(T,\Delta i)(\hat{l}_{eff})\propto\Lambda_{\mathrm{Re}\max}$ (4)

is proportional to it

$\propto$ - proportionality.

So, if we know functions like $T(\Lambda_{\mathrm{Re}\max})$ and possibly $\Delta i(\Lambda_{\mathrm{Re}\max})$ we may construct a closed algebraical model

$$\begin{cases}\Lambda_{\mathrm{Re}\max}=Max\,\mathrm{Re}\,SpecC(\hat{l})\\ \hat{l}=l_{MAX}(\Delta i(\Lambda_{\mathrm{Re}\max}),T(\Lambda_{\mathrm{Re}\max}))\end{cases}\quad(5),$$

in the single-mode approximation, where

$l_{MAX}(T,\Delta i)=\frac{d}{\Delta i(1-e^{-d\cdot\Delta T})}$ or $l_{MAX}(T,\Delta i)\cong\frac{d}{\Delta i}$

VI. (Bi)-Linear Model

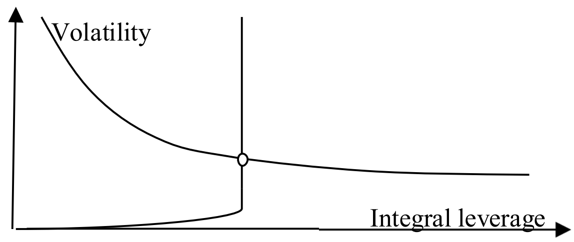

Fig. 3. The example of the supply-demand curves in coordinates of integral leverage-volatility.

At the final simplification, we escape from non-linearity and deal with tensor bi-linear equations that in the case of proper economically-connected tensors are unexpectedly easily solvable.

We introduce a modified bankruptcy formula (2')

$$l_{MAX}(T,\Delta i)=\frac{d}{\Delta i(1-e^{-d\cdot\Delta T\cdot H})}$$

that is a matter of "fantasy" at all $H\neq 1$, but for us, it is sometimes easier to solve the model at the limit

$H\to+\infty$ (6)

and then to correct the results even in several times when necessary based on the hypothesis of proportionality and the hypothesis of given T, allowing to calculate of this correction coefficient when frontally solving models like (1-3) or based on model (5).

A. Details

To fix ideas we consider open non-disturbed (or non-spatial) economics with linear finite external demand and maybe the same type supply of some goods (including part of final consumption) that in any way at very high prices transmits to infinite constant price external supply of all goods at fixed prices higher enough than (internal) general equilibrium prices. So, we practically cannot care about infinite price growth - see Fig.4

Moreover, we can think that all the internal production technologies are of the Leontievs type.

Let's define supple-demand difference components

Q_I^D - investment demand, Q_S - internal supply, Q_{IntD} - intermediate internal demand, Q_G - state final consumption, Q_C - final consumption, and Q_{NX} - net export respectively.

$$C = \frac{\partial}{\partial \vec{p}}(Q_I^D - Q_S + Q_{IntD} + Q_G + Q_C + Q_{NX}) =$$
$$= C_I^D - C^S + C^{IntD} + C^G + C^C + C^{NX}$$

The third key assumption is that in a multi-dimensional case everything stays exactly as it is shown in Fig.4: even if no investment is made $Q_I^D = 0$ total demand is higher than the maximal possible supply at full loading of internal capacities. That guarantees that at $H \to +\infty$ we have a very short crisis of time length $T \to +0$ (otherwise it could be a limited cycle of finite time length). Thus, we further work at the critical limit

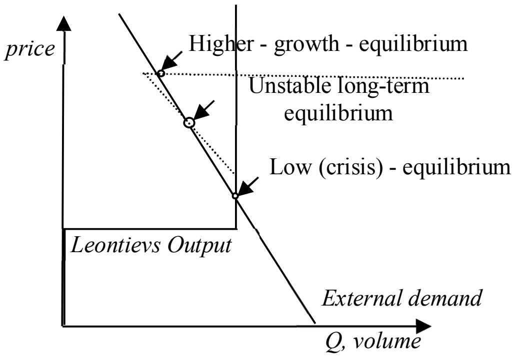

Fig. 4. The demand-supply difference in the case when the external demand is separated.

and we write that $C\Delta\vec{p} - 0 \cdot \Delta\vec{p} = \vec{0}$ - that means there exists a direction in which supply and demand coincide.

And using $J = \left(\frac{\partial \vec{i}}{\partial \vec{p}}\right)$ as a $\vec{i}$ sensitivity matrix

we may write $\Delta\vec{i} = J\Delta\vec{p}$, that defines $\begin{cases} C\Delta\vec{p} - 0 \cdot \Delta\vec{p} = \vec{0} \\ \left[\vec{l}_C^{\,*}\right] = \left[\vec{d}\right]\left[(J\Delta\vec{p})^{-1}\right] \end{cases}$,(8)

where $C = C_{00} + \sum_k C_{0k}K_k + \sum_k C_{1k}K_k l_k$ and the 2nd equation is a condition of simultaneous investment switch off. The last one means that everybody rises his leverage until he cannot hold it at the deepest crisis point and then stops to rise as $\Delta i = d / l$ leverage rises condition (see the level in Fig.1).

And if we know (from the experiment) the actual non-zero time length, we may write for actual leverage recalculation:

$$\vec{l} = \vec{l}_c \frac{1}{1 - \exp(-dT)} \quad (9)$$

Now let's make a complete solution for Leontievs production functions with A-the current expenses matrix, B-investment ones, $\vec{k}_p$ -productivity coefficients, $p_K^{\ Eq}$ -capital prices. Then

$$J = [\vec{k}_p][p_K^{\ Eq}]^{-1}(E - A)$$
$$C^S = C^{IntD} = O,$$
$$C = C_I^D + C^{IntD} + C^S + C^G + C^C + C^{NX} \cong B^T[p_K^{\ Eq^{-1}} l k_p K] \times$$
$$\times (E - A) - B^T[p_K]^{-2} B[p_K^{\ Eq} K(i_C + d)] + C_{passive} \quad ,$$

where we denote $C_{passive} = C^G + C^C + C^{NX}$. Finally, (8) transmits to

$$\begin{cases} \{B^T[p_K^{\ Eq^{-1}} l_c k_p K](E - A) - B^T[p_K]^{-2} B[p_K^{\ Eq} K(i_C + d)] + C_{passive}\}\Delta\vec{p} = 0 \\ [l_c][\vec{k}_p][p_K^{\ Eq}]^{-1}(E - \hat{A})\Delta\vec{p} = \vec{d} \end{cases}$$

after a bit unobvious solution using calibration (9) we get

$$i = i_C \vec{l}^{\,-1} = [\vec{d}]^{-1}\{[\vec{k}_p][p_K^{\ Eq}]^{-1}(E - \hat{A}) \times$$
$$\times \{B^T[\vec{p}_K^{\ Eq}]^{-2} B[p_K^{\ Eq} K(i_C + d)] - C_{passive}\}^{-1} B^T[\vec{K}][\vec{d}]\}[(1 - e^{-dT})]i_C \quad (10)$$

as a physical capital price at our single mode approximation (a ratio i_C to leverage - or its product to $\vec{l}^{\,-1}$).

If we consider state regulation when debt restructuration due to countercyclical bank rate and non-stationary inflation affects the $J \neq [\vec{k}_p][p_K^{\ Eq}]^{-1}(E - A)$ matrix, which should be calculated in a more complex way, and countercyclical state demand affects the $C^G = \partial\vec{Q}_G / \partial\vec{p}$ matrix hence $C_{passive}$.

Conclusion

So, we have formulated a common model of dynamics of debt capital and prices (1-3), based on Nash equilibrium strategies $(\hat{\vec{l}}, \hat{\vec{c}})$ defined in (1). This model was recognized as inconvenient for qualitative research at least. So we introduced a model (8) in critical approximation $H \to +\infty$ and (9) that permits us to return to a natural situation $H = 1$. This model depends only on 2d matrix J and 3d tensor C $C = \frac{\partial}{\partial \vec{p}} \Delta\vec{Q}(\vec{p}, \vec{\vartheta}, \vec{l}_{12}, \vec{K}) = C_{00} + \sum_k C_{0k}K_k + \sum_k C_{1k}K_k l_k$ and has the exact analytical solution for $\vec{l}^{\,-1}$ & i at only critical assumption, we somehow know single-period crisis longitude T.

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