A Non-Phenomenological Altman-Type Model

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***Abstract*—We consider a situation when due to a crisis a deficit of credit ability of banks arises while real sector enterprises still have to refinance their comparatively short-term debt having very high current capital return rate while total capital return rate might simultaneously be negative and substantially negative. So, we have scarce supply and considerably inelastic demand with very high-level (nearly)horizontal saturation due to the high current capital return rate. That may easily blow up the current bank rate, pushing real sector enterprises to bankruptcy. All that produces a parametrically dependent discrete mapping that depends on the same parameters as the Z-score classical Altman model. Parametrical surfaces of bifurcations of this mapping may be considered as some analog of fixed Z-score surfaces, while the corresponding scoring index, demonstrates proper dependence on the long-term return rate, debt-to-equity (debt-to-asset) ratio, and current capital share.**

Keywords—Altman model, bankruptcy, collateral equilibrium, volatility, general noise equilibrium, leverage, leverage oscillation.

I. Introduction

The article considers the possibility of a scenario in which interest rates are overestimated, which results in enterprises of the economy finding themselves in a financially risky situation, which can provide corresponding high interest rates due to high risk in a relatively long (medium) period. It is assumed that the refinancing time τ is much less than the inverse equipment depreciation rate d^{-1}. If bankruptcies are, for example, of Poisson-flow nature, then the time to bankruptcy t_b on average is the inverse intensity of this Poisson flow $\frac{1}{r}$. By estimating the time to bankruptcy, we can estimate the risk percentage $r = \frac{1}{t_b}$ and thus construct a risk-to-risk self-mapping.

The study of this mapping, as well as its modification, which takes into account the return on working capital as the limit of the market interest rate, leads to restrictions on the minimum long-term profitability, the fulfillment of which is a condition for the non-occurrence of a crisis (or the rarity of the latter, depending on a certain threshold). The right-hand side of the inequality depends on the share of equity in assets, the share of fixed capital in total capital, and the rate at which funds are disposed of.

II. The Designations Used

Indexes

^{us} – (unstable) intermediate equilibrium,

0 – initial value.

$\hat{x}$ - normalization of the value of x to d-the rate of depreciation. All values used in the article with this index are dimensionless.

Model variables

θ – the amount of debt based on the price of real (productive assets), dimensionless

$\theta = \frac{D}{p_k K}$ (1),

where D -debt, K - capital, p_k - price of capital.

$\bar{\theta} = 1 - \theta$ – equity share in capital, dimensionless. Variables θ and $\bar{\theta}$ are slow,

r - risk measured as a percentage of bankruptcy (per year) [$\frac{1}{year}$], - here and consequently we write the non-trivial dimension in brackets.

$\hat{r} = \frac{r}{d}$ - risk normalized to the rate of depreciation of fixed assets (r - fast variable in the model).

b - [$\frac{1}{year}$], bank interest (risk), b_τ - the risk bank interest smoothed over the refinancing τ period.

$b = b_0 + r$ - risk percentage, where b_0 - risk-free percentage: by default, we count $b_0 = 0$.

Main parameters:

$\sigma = \frac{i}{d}$ (or $\sigma = \frac{i_F}{d}$) - dimensionless profitability, [dimensionless] or [1].

τ - debt refinancing time - the time for which loans are taken, because $\tau << d^{-1}$ it is much less than the payback time, so loans have to be renewed,

d – rate of disposal of fixed production capital [$\frac{1}{year}$]

i_F - return on equity, [$\frac{1}{year}$],

i_{ob} - working capital return, [$\frac{1}{year}$].

ω – the proportion of working capital in the total amount of capital, dimensionless.

$$\varpi = 1 - \omega$$

- the proportion of long-term investments in the amount of capital, dimensionless.

s is the distance from the current debt level θ to the bankruptcy threshold (the latter can be designated as the level $\theta = 1$), [dimensionless].

III. Mathematical Model.

The main object of consideration will be the normalized debt (the debt to total asset ratio) θ,

$\theta = \frac{D}{p_k K}$ normalized by the amount of funds and the risk of bankruptcy associated with the possibility of debt exceeding a certain threshold close to 1:

$$\theta = 1$$

Differentiating normalized debt by time

$$\frac{d}{dt}\theta = \frac{D}{p_k K}\left(\frac{\dot{D}}{D} - \frac{\dot{K}}{K}\right)$$

$$\frac{d}{dt}D = Db_\tau - Kp_K(i_F + d)$$

where

Db_τ - debt-increasing due to the interest,

$Kp_K(i_F + d)$ - current receipts cleared of expenses, the latter follows from

$$F = Kp_K(i_F + d)$$

$$\frac{d}{dt}K = -d \cdot K \quad (2)$$

- the equation of funds' depreciation in the absence of investment:

If an investment is non-zero $I \neq 0$,

$$\frac{d}{dt}D = Db_\tau - D(i_F + d) + I$$

and

$$\frac{d}{dt}K = -d \cdot K + I/p_k,$$

However, this section deals with the case when an enterprise, realizing the threat of bankruptcy, when such a threat exists, pays off its debts as quickly as possible:

$$I = 0.$$

As a result, generalizing the formulas (40(1), ((43- 45) we get:

$$\frac{d}{dt}\theta = \frac{D}{p_k K}\left(b\frac{D}{D} + \frac{Kd}{K}\right) - \frac{D}{p_k K}\frac{(i_F + d)p_k K}{D} =$$
$$= \theta(b + d) - \frac{D}{p_k K}\frac{(i_F + d)p_k K}{D} = \theta b - (i_F + d) + \theta d$$

In total, the dynamics of specific debt θ excluding investments is described by a simple differential equation:

$$\frac{d}{dt}\theta = \theta b_\tau - (i_F + d) + \theta d \qquad (3)$$

On the right-hand side, the first term corresponds to an increase in debt due to interest payments θb , the second to income from economic activity that reduces debt, and the third θd to a redistribution of debt to a smaller amount of fixed assets due to the disposal of the latter.

We can define the risk percentage r as the inverse of the time of bankruptcy

$$r = \frac{1}{t_b},$$

which occurs when the interest rate is too high, when the debt reaches the threshold $\theta = 1$, i.e. $t_b : \theta(t_b) \tilde{=} 1$.

In fact, the threshold value can be taken as:

$$\theta(t_b) = 1 + \sigma \; ,$$

where

$\sigma = i_F / d$, which corresponds to the loss of the ability to pay debts from current income, at least at a risk-free interest rate.

Justification for the above risk assessment:

If the bankruptcy process was a Poisson flow of intensity events r , then the time before the bankruptcy occurs would be described $t_b = \frac{1}{r}$ from where it follows $r = \frac{1}{t_b}$.

The result is a discrete display of the form:

$$r_{n+1} = \frac{1}{t_b(r_n)} \qquad (4),$$

where the risk depends on the time of bankruptcy affected by the risk

The rest of the text is devoted to the issues of its research and the resulting theoretical consequences.

Here we offer two options for calculating the time of bankruptcy $t_b(r)$ – approximate and accurate. The exact version is given in Appendix 3, but for many reasons, a simpler model is used here:

Let's assume that the rate of debt growth is constant, always equal to the value that occurs at the beginning of the trajectory (i.e., $v = \theta b - (i_F + d) + \theta d$), hence we consider it approximately constant for how long the debt will grow to the threshold $\theta(t) = 1 + \sigma$. Thus, the amount of debt will have to go a distance

$s = \theta(t_b) - \theta = 1 + \sigma \ - \theta$: (see Figure 2.2).

Travel time is the distance per speed:

$$t_b = {}^{s}\!/_{v}\,.$$

As a result, we get

$$t_b = \frac{1 + \sigma_0 - \theta}{\theta b_\tau - (i_F + d) + \theta d} \qquad (5)$$

Remembering that

$$r = \frac{1}{t_b}$$

And considering that the bank rate consists of risk

$$r = b$$

we get

$$r = \frac{\theta b_\tau - (i_F + d) + \theta d}{1 + \sigma_0 - \theta} \qquad (6),$$

or

$$r = \frac{\theta r_\tau - (i_F + d) + \theta d}{1 + \sigma_0 - \theta} \qquad (7),$$

This can be considered as a mapping of risk to itself. It is linear. Given

$$\sigma = \frac{i}{d} \qquad (8),$$

we get

$$r = \frac{\theta r_\tau / d - (\sigma + 1) + \theta}{1 + \sigma_0 - \theta} d \qquad (9)$$

and, by renormalizing $\hat{r} = \dfrac{r}{d}$, finally

$$\hat{r} = \frac{\theta}{1 + \sigma_0 - \theta} \hat{r} - 1 \qquad (10),$$

Written as a mapping, we have

$$\hat{r}_{n+1} = \frac{\theta}{1 + \sigma_0 - \theta} \hat{r}_n - 1\,.$$

We are modifying it to exclude the possibility of negative risk (see Fig 1):

$$\hat{r}_{n+1} = \max(0, \frac{\theta}{1 + \sigma_0 - \theta} \hat{r}_n - 1) \qquad (11)$$

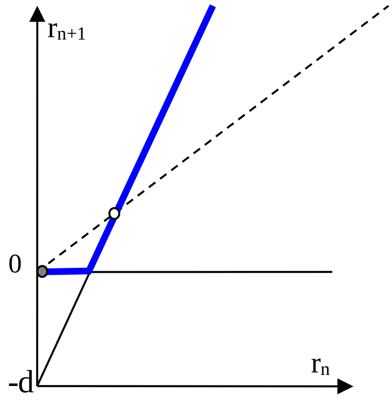

Fig. 1. Graph and equilibrium mapping of risk to itself

$$\hat{r}_{n+1} = \max(0, \frac{\theta}{1 + \sigma_0 - \theta} \hat{r}_n - 1)\,.$$

Percentage saturation.

In the short term, it is profitable to borrow as long as the interest and risk do not exceed the return on working capital. Its profitability $r_{\max}$ can serve as the upper limit of saturation

$$r_{\max} = ((\sigma \ + 1)\frac{1}{\omega} - 1)d \qquad (12),$$

or

$$\hat{r}_{\max} = (\sigma \ + 1)\frac{1}{\omega} - 1 \qquad (13)$$

From this, we get the following type of risk mapping: (see Fig 2)

$$r_{n+1} = \min\left(\max(0, \frac{\theta}{1 + \sigma_0 - \theta} r_n - d), r_{\max} \right) \qquad (14).$$

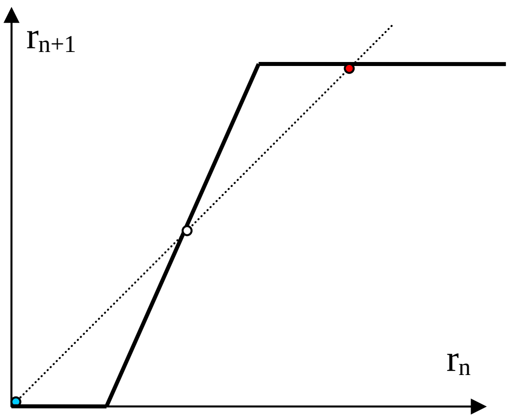

Fig. 2. Graph and equilibrium mapping of risk to itself

$$r_{n+1} = \min\left(\max(0, \frac{\theta}{1 + \sigma_0 - \theta} r_n - d), r_{\max} \right).$$

Similarly, you can formulate a piecewise given form for accurate calculation of the lifetime

$$r_{n+1} = \min\left(\max\left(0, \frac{\hat{r}_n + 1}{\ln \frac{(\hat{r}_n + 1) - (\sigma_0 + 1)}{\theta_0 (\hat{r}_n + 1) - (\sigma_0 + 1)}} d \right), r_{\max} \right) \quad (15),$$

calculated by solving a differential equation.

Piecewise linear mapping is certainly easier to study, and a full study is given below only for this approximate representation (Figure 2.3):

$$\hat{r}_{n+1} = \min\left(\max(0, \frac{\theta}{1+\sigma - \theta} \hat{r}_n - 1), \hat{r}_{\max} \right) \quad (16)$$

Depending on the combination of parameters, the curve of this mapping can have from 1 to 3 equilibria.

The appearance of a crisis (upper) equilibrium corresponds to the fold bifurcation, which occurs when

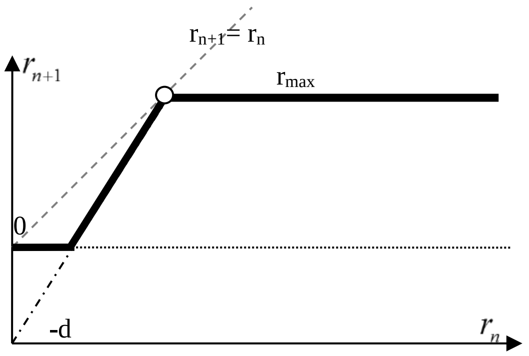

Fig. 3. Risk-to-risk mapping – bifurcation of bad (upper) equilibrium born.

$r_{us} = r_{\max}$ (see Fig.3):

$$r_{us} = \frac{1+\sigma_0 - \theta}{2\theta - 1 - \sigma_0} d$$

The moment of appearance of the upper equilibrium

$$r_{us} = r_{\max} \quad (17),$$

where is the unstable equilibrium

$$\hat{r}_{us} = \frac{1+\sigma - \theta}{2\theta - 1 - \sigma} \quad (18)$$

there is a solution to the equation

$$\hat{r}_{n+1} = \hat{r}_n$$

For an oblique-middle section of the display.

(it takes the form $\hat{r} = \dfrac{\theta}{1+\sigma - \theta} \hat{r} - 1$).

The equation

$$\hat{r}_{us} = \frac{1+\sigma - \theta}{2\theta - 1 - \sigma} \quad (18)$$

takes the form

$$(\sigma + 1)\frac{1}{\omega} - 1 = \frac{1+\sigma - \theta}{2\theta - 1 - \sigma} \quad \Leftrightarrow$$

$$(\sigma + 1)\frac{1}{\omega} - 1 = \frac{\theta}{2\theta - 1 - \sigma} - 1 \quad (19),$$

What generates a quadratic equation

$$(\sigma + 1)(2\theta - 1 - \sigma) = \theta\omega \quad (20)$$

Having a larger solution

$$\sigma + 1 = \theta + \sqrt{\theta^2 - \omega\theta} \quad (21)$$

Being interpreted as an inequality describing the region of non-occurrence of bad upper equilibrium, it takes the form

$$\sigma + 1 > \theta + \sqrt{\theta^2 - \omega\theta} \quad (22)$$

$\sigma = 0$ This condition is particularly easy to write down:

$$\omega > 2 - \frac{1}{\theta}$$

(see the upper hyperbolic line in the Fig.4)

Traditionally, we will consider the main equilibrium to be the one with the higher potential, which is responsible for the ratio of probabilities of being in both equilibria.

The equality of probabilities of being in upper and lower equilibrium corresponds to

$$\int_0^{\hat{r}_{\max}} \tilde{f}(\hat{r}) d\hat{r} = 0,$$

which means

$$U(\hat{r}_{\max}) = U(0)$$

(generally, it is more correct to say that the probability densities in the maxima of the distribution $\rho(r)$ corresponding to the equilibria are equal).

Thus, for equality of potentials, it is necessary to have equal areas of figures between the display graph and the diagonal. In the case of a 3-term piecewise linear map, the first and third components of which are constants

Specifically, in the case of mapping

$$r_{n+1} = \min\left(\max(0, \frac{\theta}{1+\sigma - \theta} r_n - d), r_{\max} \right) \quad (23)$$

and whether in the form of a differential equation

$$\frac{dr}{dt} = \min\left(\max(0, \frac{\theta}{1+\sigma - \theta} r - d), r_{\max} \right) - r \quad (24)$$

this corresponds to finding an unstable equilibrium exactly in the middle - at the level of half the height of the crisis equilibrium

$$2 r_{us} = r_{\max} \quad (25)$$

and whether

$$(\sigma + 1)\frac{1}{\omega} - 1 = 2\frac{1+\sigma - \theta}{2\theta - 1 - \sigma} \quad (26)$$

$$\frac{\sigma+1}{\omega}-1=\frac{2+2\sigma-2\theta}{2\theta-1-\sigma} \qquad (27)$$

$$\frac{\sigma+1}{\omega}-1=\frac{\sigma+1}{2\theta-1-\sigma}-1 \qquad (28)$$

or

$\omega=2\theta-1-\sigma$,

which results in a threshold

$$\sigma = 2\theta-1-\omega \qquad (29)$$

or

$$\sigma \ge 2\theta-1-\omega \qquad (30)$$

The latter can be rewritten as

$$\sigma \ge (1-\omega)-2(1-\theta) \qquad (31)$$

Expressions in parentheses are interpreted as equity $\bar{\theta}=1-\theta$ and share of long-term investments $\varpi=1-\omega$.

Thus

$$\sigma>\varpi-2\bar{\theta} \qquad (32),$$

If $\sigma=0$ profitability is zero, the amount of equity should cover at least half of long-term investments.

$$\bar{\theta}>\frac{1}{2}\varpi \qquad (33)$$

(see the straight line from the up-right corner in the Fig.4)

Return on equity (return on equity at the risk-free loan rate)

$$i_{own}=\frac{1}{1-\theta}i_F \qquad (34)$$

Limited by the value of

$$i_{own}=\frac{2}{\varpi-\sigma}i_F=2\frac{d\cdot\sigma}{\varpi-\sigma} \qquad (35),$$

The risk-return rate of the own capital has the form

$$i_{own}^{risk}(\theta)=\frac{2}{1-\theta}(i_F-r(\theta))=\frac{2}{1-\theta}(i_F-r(\varpi,\theta)) \qquad (36)$$

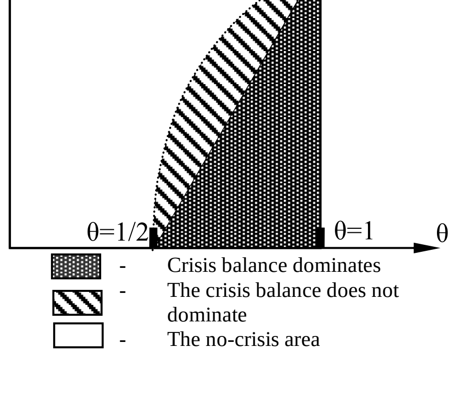

Fig. 4. Parametric boundaries of crisis impossibility zones, predisposition zone, and intermediate zone at zero IRR section.

The same inequality of leverage

$$\sigma>\varpi-\frac{2}{l} \qquad (37)$$

$l<\frac{2}{\varpi-\sigma}$ by $\varpi-\sigma>0$

This model can be compared with the Altman model.

$$Z=0{,}717\omega+(0{,}847+3{,}107)i_F+3{,}107d+0{,}42\frac{1-\theta}{\theta}+0{,}995\frac{i_F}{1-\theta} \qquad (38)$$

where Z is a measure of proximity to bankruptcy.

In the original coordinates

$$Z=0{,}717A+0{,}847B+3{,}107C+0{,}42D+0{,}995E\,, \qquad (39)$$

where the letters denote the following dimensionless ratios:

A – working capital/ amount of assets

B – non-distributed profit/ amount of assets

C – operating profit/ total assets

D – carrying amount of equity/ borrowed liabilities

E – revenue/ total assets

If we transform the inequality (32) to

$$\sigma-f=(1-\omega)-2(1-\theta),\ f\ge 0 \qquad (40)$$

We may consider f as a measure in some way anti-correlated with risk (in terms of Poisson or non-Poisson bankruptcy flow intensity) so far

$$f=\sigma-(1-\omega)+2(1-\theta)$$

or

$$f=\frac{1}{d}\,i_F+(\omega-1)+2(1-\theta) \qquad (41)$$

The same approach may be applied to (22)

$$\sigma+1>\theta+\sqrt{\theta^2-\omega\theta}$$

transiting to

$$\sigma+1-f_1=\theta+\sqrt{\theta^2-\omega\theta}$$

or

$$f=\frac{1}{d}i_F+1-\theta-\sqrt{\theta^2-\omega\theta}$$

In both cases we have a positive dependence on current capital share ω and internal return rate:

$\frac{\partial}{\partial i_F}f>0$, $\frac{\partial}{\partial i_F}f_1>0$ as well as $\frac{\partial}{\partial i_F}Z>0$

and $\frac{\partial}{\partial\omega}f>0$, $\frac{\partial}{\partial\omega}f_1>0$ as well as $\frac{\partial}{\partial\omega}Z>0$, while dependence on θ is always negative in these three cases.

IV. CONCLUSION.

We had considered a situation when a self-fulfilling forecast may lead to the rise of interest rates on the supply side while the demand side is permanently ready to take loans for refinancing the previous ones at any sensible interest rate up to the internal return rate of the current capital (which is generally radically higher than the general return rate). Yet, depending on the two substantial parameters – the initial debt-to-asset ratio and the technological share of the current capital this leads to more or less fast coming to bankruptcy and the latter circumstance is the reason the risk surplus to the interest rate may achieve this limit (the current capital return rate), that leads to a formation of curves reflecting some qualitative changes of the risk-to-risk mapping in the $(\bar{\theta}, \varpi)$ flat of parameters.

Despite the omitting in this toy-like qualitative model of nearly all factors concerning the supply side of the credit process (that should be a part of much wider consideration), we had caught at least some dependencies qualitatively similar to the classical Altman-type/Z-score experimental models.

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