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TOPICAL ISSUE

Two-Directional Traffic Scheduling Problem Solution for a Single-Track Railway with Siding

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Abstract—The paper is concerned with scheduling trains moving in both directions between two stations connected by a single-track railway with a siding. The paper presents dynamic programming based algorithms which minimizes two objective functions: maximum lateness and total weighted completion time. The complexity of these algorithms is $O(n^2)$.

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1. INTRODUCTION

The problem of optimal scheduling of trains on a single-track segment of railway is considered. There are two sets of trains, N_1 and N_2. The trains of the set N_1 moves from station 1 to station 2, the trains of the set N_2 moves in the opposite direction. There is a siding between the stations for passing the oncoming trains. It is necessary to schedule the train traffic, that is, to establish for each train the departure time and the time spent at the siding.

In practice this problem arises in railway networks for inter-department transportation in manufacturing. Train scheduling in the case of track closure in a double-track railway and the so-called bottlenecks in large railway networks exemplify other applications.

A review of models and methods of railway scheduling is given in [1–3]. The relation between the problem of scheduling on a single-track railway with the classical problems of the scheduling theory for several machines was shown in [4, 5]. This approach where the track segments play the role of machines and the trains the role of jobs, was used for example in [6–10]. It was observed that already in the case of three machines (railway segments) this problem in general is NP-hard [6].

The models of integer of linear programming for the problems with a single-track railways can be found, for example, in [11, 12]. The heuristic algorithms for the large-scaled NP-hard problems are developed, for example, in [9, 10, 13, 14]. The polynomially solvable cases having practical applications represent another direction of research to which the present paper belongs. Among the relevant publications can be mentioned [7, 8, 15–17].

The problem of train scheduling between two stations connected by a single track, divided by semaphores, is considered in [7]. It is converted into the single-machine scheduling problem with setup times for which dynamic programming based algorithms are proposed.

The case where between two stations there are sidings (intermediate stations) is considered in [8]. A pseudopolynomial time algorithm is proposed for the problem of minimizing the completion time of transportation. In the case of a single intermediate station with the capacity of one train, this problem is very close to that considered in this paper, but differs in the objective function and safety time intervals between two successive trains. The problem with more than one intermediate station of unlimited capacity is considered in [15]. This paper is concerned with computational complexity and some polynomially solvable cases. The problem of scheduling traffic on a double-track line where one of the tracks is closed for repair is considered in [16]. The trains have release times, deadlines, and different speed. Since this problem is NP-hard, the method of local search based on the polynomial algorithm for solving the problem with certain order of train departures is suggested. It is assumed that at most one train can traverse the track at a time, that is, it is impossible to have a group of trains simultaneously moving from the same station. The two-station problem with a siding also is considered in [17]. This paper presents an analytical solution for the problem of minimization of the completion time of all transportations.

The paper is organized as follows. Section 2 gives a mathematical formulation of the problem. Section 3 presents some properties of the schedules of this model. Section 4 introduces the notion of state and defines the rules for transition between states underlying the algorithms in Section 5.

2. THE PROBLEM

Consider a single track with a siding which can keep in the siding additional track one train. The siding splits the track into two segments (see Fig. 1). The segment between station 1 and the siding is called segment A (to the left of the siding in Fig. 1), and the segment between the siding and station 2, segment B (to the right of the siding in Fig. 1).

It is assumed that

- the speed of all trains is the same and is a constant;
- the traveling time for the segments A and B are p_1 and p_2, respectively, and, without loss of generality $p_1 \geqslant p_2$;
- the number of trains of the set N_1 is n_1, the set N_2 is n_2;
- each train $i \in N_s$ from station s, $s \in \{1, 2\}$, has the due date d_s^i or weight w_s^i (depends on the objective function);
- all trains are ready to depart at the initial moment;
- the minimal time between the departures of two trains from the same station and arrival of two trains to the siding are given;
- the minimal time between the arrival of a train to a station and the next departure from this station is given.

To simplify the presentation, assume that the safety intervals times are equal. Denote them by β, $\beta < p_2 \leqslant p_1$. For different safety time intervals, the approach to the problem is similar. The schedule satisfying the above constraints is called the feasible schedule.

A schedule σ specifies

- the departure time $S_s^i(\sigma)$ of each train $i \in N_s$, $s \in \{1, 2\}$;
- the time $\tau_s^i(\sigma)$ that each train spends in the siding, $i \in N_s$, $s \in \{1, 2\}$.

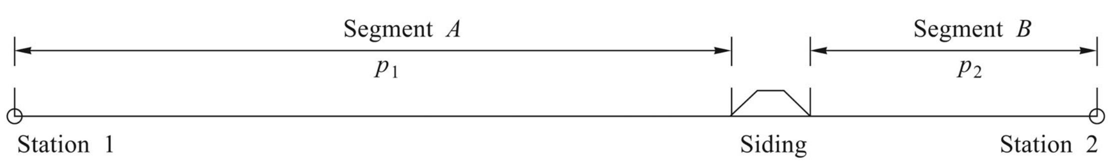

Fig. 1. Track outlay.

For schedule σ, denote by $C_s^i(\sigma)$ the arrival time of train $i \in N_s$ to destination. Then,

$$C_s^i(\sigma) = S_s^i(\sigma) + p_1 + p_2 + \tau_s^i(\sigma). \tag{1}$$

Consider the problems of minimizing the following objective functions:

the maximum lateness

$$L_{\max}(\sigma) = \max_{i \in N_s,\, s \in \{1,\,2\}} \left\{ C_s^i(\sigma) - d_s^i \right\}; \tag{2}$$

total weighted completion time

$$\sum w_i C_i(\sigma) = \sum_{i \in N_s,\, s \in \{1,\,2\}} w_s^i C_s^i(\sigma). \tag{3}$$

These objective functions are of practical importance because (2) takes into account the due times and (3) takes into account the importance of freights. According to the conventional three-position system of notation [18], the following brief notation can be introduced for the given problems:

$$S2S1|siding_i = 1, t_j = t|L_{\max},$$
$$S2S1|siding_i = 1, t_j = t|\sum w_i C_i,$$

where $S2$ denotes two stations, $S1$ denotes one siding, $siding_i = 1$ characterizes the capacity of the siding, and $t_j = t$ indicates that all trains have the same speed.

3. PROPERTIES OF SCHEDULES

Definition 1. Train $i \in N_s$, $s \in \{1, 2\}$, is called the express in schedule σ if it does not stop at the siding that is, $\tau_s^i(\sigma) = 0$.

In what follows, we consider only schedules where trains do not stop at the siding if they do not pass other trains, because a stop at the siding without an oncoming train is useless and can only increase the value of the objective function. The trains stopping at the siding are called the non-expresses.

Definition 2. A schedule is regular if it is impossible to reduce the arrival time of any train without increasing the arrival times of some other trains and without changing the order of passing the siding by trains.

Figure 2a depicts a feasible schedule, and Fig. 2b shows an example of the regular schedule. The horizontal lines represent the stations and siding, and the inclined lines represent the movement of trains. The horizontal axis shows time and the vertical axis shows coordinates. If the train stops at the siding for a certain time, then within this interval the horizontal line corresponds to this train. Circles in the figures correspond to the instants when the express passes the siding occupied by a train from another station.

We note that in the scheduling theory (see [6] for example) the class of similar schedules is regarded as that of active ones. By the active schedules are meant those where all jobs start as early as possible, that is, it is impossible to reduce the time of starting a job without changing the time of execution of other jobs. In the present paper, introduced definition is the notion of a wider class of regular schedules, because the non-expresses do not necessarily start motion at the earliest of the available times, that is, not every regular schedule is active.

The following theorem shows that search of the optimal schedules can be constrained to the set of regular schedules.

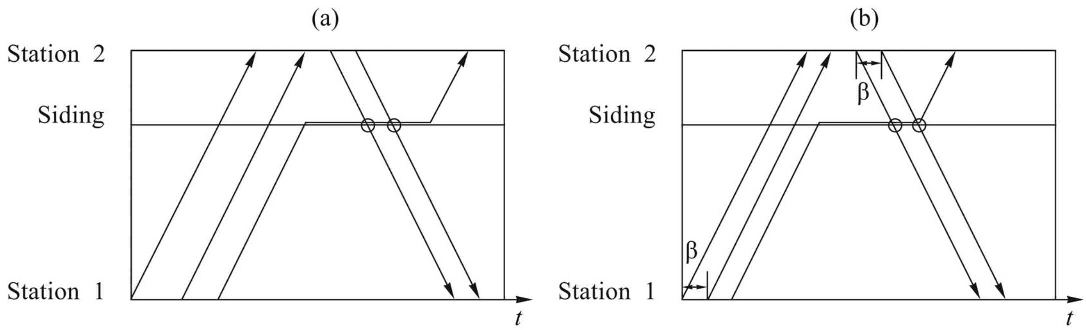

Fig. 2. Graph of trains under (a) feasible and (b) regular schedules.

Theorem. *For any feasible schedule σ there exists a regular schedule $\bar{\sigma}$ for which*

$$L_{\max}(\bar{\sigma}) \leqslant L_{\max}(\sigma), \quad \sum w_i C_i(\bar{\sigma}) \leqslant \sum w_i C_i(\sigma). \tag{4}$$

Proof. We demonstrate how by using the schedule σ one can construct a regular schedule $\bar{\sigma}$ with the value of objective function no more than under the schedule σ. Consider $2(n_1 + n_2)$ instants of trains entering the siding and leaving it in the current schedule and numerate them in the nondecreasing order. If some instant of leaving the siding by a non-express coincides with the instant of express leaving this siding, the smaller number is assigned to the time of express departure. Additionally, to the instant of express arrival we assign a number smaller than to the instant of departure of the same express. We take the instant with minimal number k such that it can be reduced by changing the time of departure of the corresponding train from the station or from the siding without changing the graphs of other trains. Reduce this instant to the minimal possible value in the current schedule. In the resulting schedule again find the instant with minimal number l that can be reduced, and show that the new instant has a greater number than the preceding one, that is, $l > k$. Let k be the instant of arrival to or departure from the siding of some train i. All trains corresponding to the $k - 1$ first instants already left the siding before time k, except for, possibly, one train passing another train number i. Therefore, shift of the moment k can't impact on their movement. If there exists the train which gives the way to train i in the siding, then its arrival time to the siding can't be reduced after the shift of k, because is moves to the siding from the other station, and it's exit moment number is more than number k. Consequently, a schedule where it is impossible to reduce any instant of train departure, that is, a regular schedule, can be obtained at most in $2(n_1 + n_2)$ steps. Since the instants of departures from the siding were not increased, the instants of train arrivals to the destination station also were not increased. Consequently the values of the objective function did not increase at modification of the schedule, which proves the theorem.

Therefore, in what follows we without loss of generality consider only the regular schedules. The following two lemmas enable us to define the order of train departures from each station.

Lemma 1. *For the problem $S2S1|siding_i = 1, t_j = t|L_{\max}$ there exists the optimal regular schedule σ for which the trains of each set N_1 and N_2 depart in the order of their nondecreasing deadlines, that is, for any trains i and j from one station s the inequality $d_s^i < d_s^j$ entails the inequality $S_s^i(\sigma) < S_s^j(\sigma)$.*

Lemma 2. *For the problem $S2S1|siding_i = 1, t_j = t|\sum w_i C_i$ there exists an optimal regular schedule σ for which the trains from each of the sets N_1 and N_2 depart in the nonincreasing order of their weight coefficients, that is, for any trains $w_s^i > w_s^j$ from the same station s the inequality $d_s^i < d_s^j$ entails the inequality $S_s^i(\sigma) < S_s^j(\sigma)$.*

These lemmas are readily proved using the popular permutation method of the scheduling theory stating that if there are two trains sent in another order, they can be interchanged without increasing the objective function (see, for example, [6, 19]).

Now we consider only the regular schedules with the orders of train departure according to Lemmas 1 and 2. We prove that the number of various regular schedules is not polynomial. For that we construct the lower estimate of cardinality of the considered set of regular schedules. Let under a regular schedule the siding be unusable, that is, the possible number of variants be reduced. Then, the number of possible regular schedules is equal to the number of permutations of the order of $n = n_1 + n_2$. The order of train departures from each station is known, therefore, the number of regular schedules is equal to $\frac{(n_1+n_2)!}{n_1!n_2!}$.

The following Lemmas 3 and 4 formulate variation of the optimal schedule at shifting the time of train departures.

Lemma 3. *Let σ_0^* be the regular optimal schedule of the solution of the problem $S2S1|siding_i = 1, t_j = t|L_{\max}$. Then, the schedule σ_r^*, where for any train $i \in N_s,\ s \in \{1,2\}$,*

$$S_s^i(\sigma_r^*) = S_s^i(\sigma_0^*) + r,$$
$$\tau_s^i(\sigma_r^*) = \tau_s^i(\sigma_0^*),$$

is the regular optimal schedule of the problem where the start time of all transportations is r. At that, the optimal values of the objective function is $L_{\max}(\sigma_0^*) + r$.

Proof. Assume that the schedule σ_r^* is not optimal in the problem with start of all transportations shifted by r. Then, there exists a feasible schedule σ_r for which all trains depart not earlier than at the instant r such that

$$L_{\max}(\sigma_r) < L_{\max}(\sigma_r^*). \tag{5}$$

Construct the schedule σ_0 where for each train $i \in N_s,\ s \in \{1,2\}$,

$$S_s^i(\sigma_0) = S_s^i(\sigma_r) - r,$$
$$\tau_s^i(\sigma_0) = \tau_s^i(\sigma_r).$$

This schedule is feasible in the initial problem $S2S1|siding_i = 1, t_j = t|L_{\max}$. In virtue of (5), at that

$$\begin{aligned} L_{\max}(\sigma_0) &= \max_{i\in N_s, s\in\{1,\,2\}} \left\{C_s^i(\sigma_0) - d_s^i\right\} = \max_{i\in N_s, s\in\{1,\,2\}} \left\{C_s^i(\sigma_r) - r - d_s^i\right\} \\ &= L_{\max}(\sigma_r) - r < L_{\max}(\sigma_r^*) - r = \max_{i\in N_s, s\in\{1,\,2\}} \left\{C_s^i(\sigma_r^*) - d_s^i\right\} - r \\ &= \max_{i\in N_s, s\in\{1,\,2\}} \left\{C_s^i(\sigma_0^*) + r - d_s^i\right\} - r = L_{\max}(\sigma_0^*), \end{aligned}$$

which contradicts optimality of the schedule σ_0^*.

Lemma 4 for the problem $S2S1|siding_i = 1, t_j = t|\sum w_iC_i$ can be proved along the same lines.

Lemma 4. *Let σ_0^* be the regular optimal schedule of the problem $S2S1|siding_i = 1, t_j = t|\sum w_iC_i$. Then, the schedule σ_r^*, where for any train $i \in N_s,\ s \in \{1,2\}$,*

$$S_s^i(\sigma_r^*) = S_s^i(\sigma_0^*) + r,$$
$$\tau_s^i(\sigma_r^*) = \tau_s^i(\sigma_0^*),$$

is the regular optimal schedule of the problem where the transportation of trains begins at r. At that, the optimal value of the objective function is equal to $\sum w_iC_i(\sigma_0^*) + r\sum_{i\in N_s, s\in\{1,2\}} w_s^i$.

4. CLASSIFICATION OF TRAINS AND STATES UNDER REGULAR SCHEDULE

Each express either goes through an empty siding or is passed by a non-express train from the other station. We introduce a definition enabling us to classify the expresses.

Definition 3. Under regular schedule σ, by the group of all expresses leaving successively one station at the interval β is meant

(i) "batch with empty siding line" if expresses go through empty siding,

(ii) "batch with occupied siding" if expresses go through siding occupied by a train from another station.

Figure 3 depicts a train traffic graphs corresponding to the batches with empty and occupied siding depending on the express departure station.

On the basis of the above variants we introduce the notion of express type.

Definition 4. By the express type under regular schedule σ we mean a set of two parameters (s, b), the parameter s defines the station from which the express departs, that is, $s \in \{1, 2\}$ and b takes the value:

- "0" if express goes in a batch with empty siding,
- "1," "2" if express goes in a batch with occupied siding place:
  —"1" if the given express is not the last in the batch,
  —"2" if the given express is the last in the batch.

Figure 4 shows the graphs of expresses of different types:

- express goes in a batch with empty siding (type $(s, 0)$);
- first in a batch with occupied siding (type $(s, 1)$);
- in the middle of batch (type $(s, 1)$);
- and last in the batch (type $(s, 2)$).

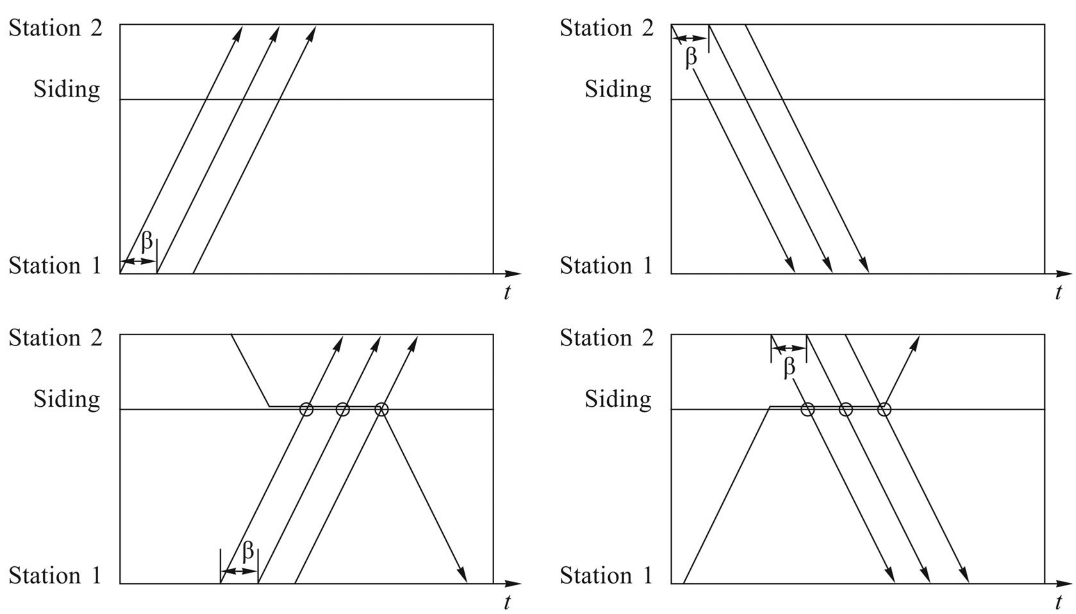

Fig. 3. Graphs of train traffic for batches with empty and occupied sidings.

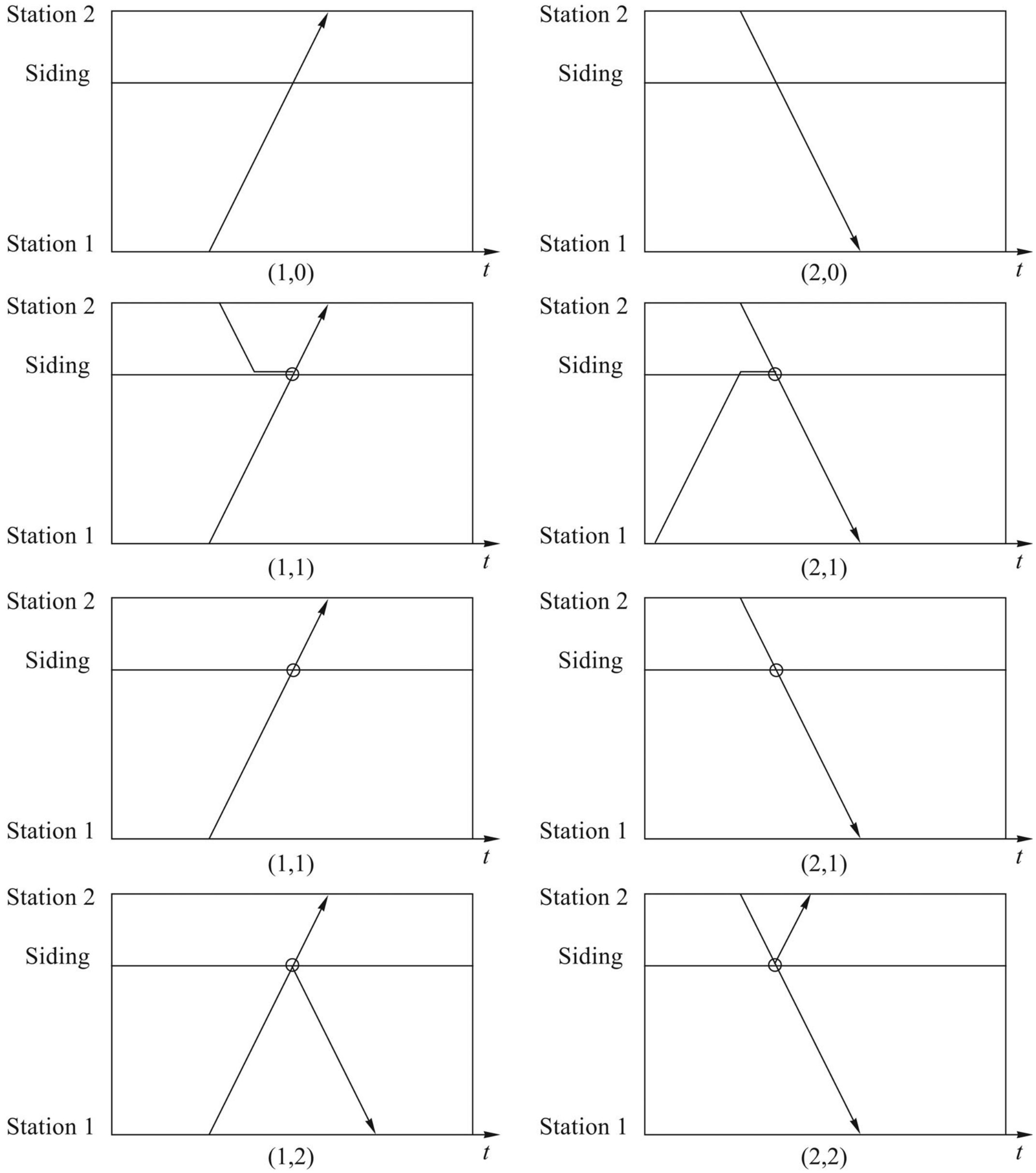

Fig. 4. Graphs of expresses of different types.

Each schedule defines its set of expresses and the order of departure of these expresses. Therefore, all expresses in the schedule can be ordered by the times of their departures from or arrivals to the destination stations. On the set of express types introduce the binary relation "$\rightarrow$" representing the feasible pairs of types of two successive expresses. If an express of type (s', b') can depart immediately after an express of the type (s, b), we denote this fact as $(s, b) \rightarrow (s', b')$. By $\bar{s}$ we denote the number of station opposite to the station with number $s \in \{1, 2\}$. Define the possible pairs of types of successive expresses.

Lemma 5. *Let (s, b) and (s', b') be the types of two successive expresses in the regular schedule σ. Then, the following combinations of pairs (s, b) and (s', b') are possible:*

- *if $b \in \{0, 2\}$, then $(s', b') \in \{(s, 0), (\bar{s}, 0), (s, 1), (\bar{s}, 1), (s, 2), (\bar{s}, 2)\}$;*
- *if $b = 1$, then $(s', b') \in \{(s, 1), (s, 2)\}$.*

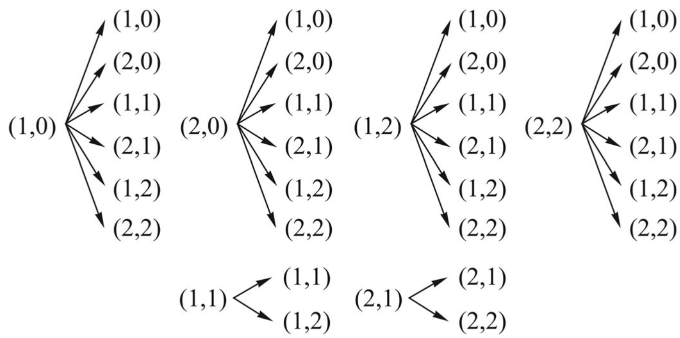

Fig. 5. Possible pairs of types of successive expresses.

Proof. The proof of Lemma 5 follows directly from the definitions of batches, types of expresses and regular schedule. If the first express in the pair of successive trains has type $(s, 0)$, that is, passes an empty siding , or type $(s, 2)$, that is, completes the batch, then the following express can have an arbitrary type. If the batch with occupied siding is not completed, that is, the express has type $(s, 1)$, then the following express can be only of type $(s, 1)$ or $(s, 2)$. Indeed, in this case the preceding express did not complete the batch, and, therefore, in virtue of regularity of the schedule the non-express making way for it is still at the siding. Consequently, the following express cannot go from the opposite station. For the same reason, the next express cannot be of type $(s, 0)$ because the siding is occupied.

All possible combinations of the types of expresses are shown in Fig. 5.

The order of train departure from each station is known by virtue of Lemmas 1 and 2. We numerate the trains at each station in the order inverse to the order of their departure, that is, for any trains i and j from one station s in any schedule σ

$$i < j \text{ entails inequality } S_s^i(\sigma) > S_s^j(\sigma). \tag{6}$$

For each $k_1 \in \{0, 1, \dots, n_1\}$ and each $k_2 \in \{0, 1, \dots, n_2\}$, we determine the sets of trains $K_1 \subseteq N_1$ and $K_2 \subseteq N_2$: $K_s = \{1, \dots, k_s\}$ for $k_s > 0$ and $K_s = \emptyset$ for $k_s = 0$, $s \in \{1, 2\}$.

Definition 5. Let $k_1 \in \{0, 1, \dots, n_1\}$, $k_2 \in \{0, 1, \dots, n_2\}$, $s \in \{1, 2\}$, $b \in \{0, 1, 2\}$, $k_s > 0$ and $k_{\bar{s}} > 0$ for $b > 0$. By the subproblem $\mathcal{P}(k_1, k_2, s, b)$ is meant the problem where

- it is necessary to bring the trains of the sets K_1 and K_2 to the destinations;
- the first express departs at time $t = 0$ and has type (s, b);
- if $b > 0$, then by the time of arrival of the first express the train from station $\bar{s}$ with the number $k_{\bar{s}}$ is already at the siding.

The set of parameters (k_1, k_2, s, b) is called the state corresponding to the given subproblem.

Since in virtue of Lemmas 1 and 2 the orders of departures from each station are known, the state (k_1, k_2, s, b) defines uniquely the number of express k_s departing first and its type, as well as the number of the non-express $k_{\bar{s}}$ which lets it pass (for $b \neq 0$).

We determine the system states that can occur successively:

(1) in the initial state, no one train was delivered;

(2) the last express either goes through the siding or is the last in the batch with occupied siding;

(3) two successive states must correspond to an admissible pair of types of successive expresses, that is, conditions of Lemma 5 are satisfied for them;

(4) if in the preceding state the express from station s completed the batch with occupied siding, then in the next state it is required to deliver to each station one train less (delivered are express and non-express from the opposite stations).

In the other cases, in the subsequent state at the station from which the express arrived the number of trains is decreased by one. This can be defined formally as follows.

Definition 6. The sequence of states

$$(k_1^1, k_2^1, s^1, b^1), \ldots, (k_1^e, k_2^e, s^e, b^e),$$

where e is the number of expresses, is called admissible if:

1) $k_1^1 = n_1,\ k_2^1 = n_2$;

2) $(k_1^e, k_2^e, s^e, b^e) \in \{(1,0,1,0), (0,1,2,0), (1,1,1,2), (1,1,2,2)\}$;

3) for any $i \in \{1, \ldots, e-1\}$ $(s^i, b^i) \to (s^{i+1}, b^{i+1})$;

4) for any $i \in \{1, \ldots, e-1\}$

$$k_{s^i}^{i+1} = k_{s^i}^i - 1, \quad k_{\bar{s}^i}^{i+1} = \begin{cases} k_{\bar{s}^i}^i - 1 & \text{if } b^i = 2 \\ k_{\bar{s}^i}^i, & \text{otherwise.} \end{cases} \tag{7}$$

Any admissible sequence of states corresponds to some regular schedule.

5. DYNAMIC PROGRAMMING-BASED ALGORITHM

The algorithm based on the method of dynamic programming solves successively the subproblems with all admissible values of k_1 and k_2 beginning from the minimal values ($k_1 = 1$ and $k_2 = 0$, $k_1 = 0$ and $k_2 = 1$, $k_1 = 1$ and $k_2 = 1$) and ending with the values $k_1 = n_1$ and $k_2 = n_2$.

Each state characterizes a situation where an express departs from a station. At that, under the given order of express departures in the regular schedule the time of departure of each express depends only on the types of expresses. This allows one to consider the system only at the departure moments of expresses. Define the function $h((s,b),(s',b'))$ as the difference between the departure moments of two successive expresses of types (s,b) and (s',b') in the regular schedule. Lemmas 6–10 give the possible values of the function $h((s,b),(s',b'))$.

Lemma 6. *For each of the following pairs of types of successive expresses* $((s,0),(s,0))$; $((s,1),(s,1))$; $((s,1),(s,2))$, *the difference between their departure moments is given by*

$$h((s,b),(s',b')) = \beta.$$

Proof. In all three cases both expresses depart from the same station within one batch. In virtue of the schedule regularity the interval between their departure moments is equal to β.

Lemma 7. *For a pair of types of the successive expresses* $((s,2),(s,0))$, *as well as for* $2(p_1 - p_2) \geqslant \beta$, *for the pairs* $((1,2),(1,1))$ *and* $((1,2),(1,2))$ *the difference between the express departure moments is equal to*

$$h((s,b),(s',b')) = 2p_s + \beta.$$

Proof. 1. Let the successive expresses have types $(s,2)$ and $(s,0)$. The first express completes the batch. Consequently, the non-express making way for it departs in time p_s after the departure of the first express from the siding and the segment of track between the siding and station s will be occupied for additional p_s time units. In time β after the arrival to station s of the non-express,

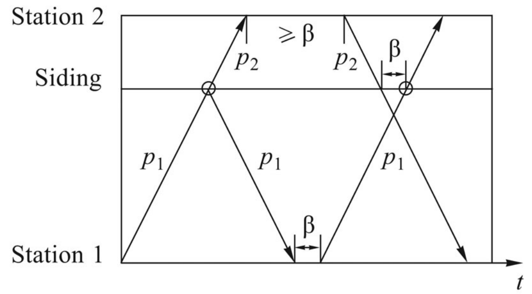

Fig. 6. Illustrated difference between the departure moments of the successive expresses of types $(1, 2)$ and $(1, 1)$.

the next express of type $(s, 0)$ departs. As the result, we get the inter-express interval equal to $p_s + p_s + \beta$.

2. Let $2(p_1 - p_2) \geqslant \beta$. Consider a pair of successive expresses of types $(1, 2)$ and $(1, 1)$. In time p_1 after the departure of the first express, the siding is left by the passing non-express, and the segment of path between the siding and station 1 is occupied p_1 time units more. Consequently, the second express can leave the first station not earlier than after $2p_1 + \beta$ time units. We prove that if the second express departs precisely after $2p_s + \beta$, then the non-express passing it at the siding has the time to reach the siding before β before the express. Indeed, in this case the second express reaches the siding in $2p_1 + \beta$ after the departure of the first express, that is, the non-express giving way to it should reach the siding in $2p_1$ after the departure of the first express. Taking into consideration that the non-express has to wait for the arrival to station 2 of the first express (time p_2), wait during the safety interval β and reach the siding (time p_2). Therefore, it reaches the siding not earlier than at time $2p_2 + \beta$. Then, for $2p_1 \geqslant 2p_2 + \beta$ the non-express has the time to reach the siding (see Fig. 6). So, the second express can leave with the interval $2p_1 + \beta$ after the first express. The proof for the pair $(1, 2)$ and $(1, 2)$ follows the same lines.

Lemmas for other values of $h((s, b), (s', b'))$ are proved in a similar manner.

Lemma 8. *For each of the following pairs of the types of successive expresses* $((s, 0), (\bar{s}, 0))$; $((s, 0), (\bar{s}, 1))$; $((s, 2), (\bar{s}, 0))$; $((s, 0), (\bar{s}, 2))$, *as well as for* $2(p_1 - p_2) \geqslant \beta$ *for the pairs* $((2, 2), (1, 1))$; $((2, 2), (1, 2))$, *the difference between the departure moments of expresses is given by*

$$h((s, b), (s', b')) = p_1 + p_2 + \beta.$$

Lemma 9. *For each of the following pairs of types of the successive expresses* $((s, 0), (s, 1))$; $((s, 0), (s, 2))$; $((2, 2), (2, 1))$; $((2, 2), (2, 2))$, *as well as under* $2(p_1 - p_2) < \beta$ *for the pairs* $((1, 2), (1, 1))$ *and* $((1, 2), (1, 2))$, *the difference between the departure moments of expresses is given by*

$$h((s, b), (s', b')) = 2(p_{\bar{s}} + \beta).$$

Lemma 10. *For each of the following pairs of types of the successive expresses* $((1, 2), (2, 1))$; $((1, 2), (2, 2))$, *as well as under* $2(p_1 - p_2) < \beta$ *for the pairs* $((2, 2), (1, 1))$ *and* $((2, 2), (1, 2))$, *the difference between the departure moments of expresses is given by*

$$h((s, b), (s', b')) = 3p_s + 2\beta - p_{\bar{s}}.$$

We describe an algorithm to construct solution for the objective function $L_{\max}$. Denote by $F(k_1, k_2, s, b)$ the optimal value of the objective function $L_{\max}$ of the subproblem $\mathcal{P}(k_1, k_2, s, b)$ on the given set of trains k_1, k_2 and given type of the first express (s, b). We recall that at each station the trains are numerated in the order inverse to the order of departures (see (6)).

Calculations proceed from the states with minimal number of undelivered trains:

$$F(1,0,1,0) = p_1 + p_2 - d_1^1;$$
$$F(0,1,2,0) = p_1 + p_2 - d_2^1;$$

$$F(1,1,1,2) = \max \begin{cases} 2p_1 - d_2^1 \\ p_2 + p_1 - d_1^1; \end{cases}$$

$$F(1,1,2,2) = \max \begin{cases} 2p_2 - d_1^1 \\ p_2 + p_1 - d_2^1. \end{cases}$$

Taking into consideration Definition 6, eliminate the unallowable combinations. For that, the values of the following subproblems are assumed to be ∞:

- $F(0, k_2, 1, 0) = \infty$;
- $F(k_1, 0, 2, 0) = \infty$;
- $F(k_1, k_2, s, b) = \infty$ for $b \neq 0$ if $k_1 = 0$ or $k_2 = 0$.

Let an express $i \in N_s$ of type (s, b) leave the departure station at the time moment $t = 0$. At that, the trains k_1 and k_2 are not yet delivered from the first and second stations, respectively, that is, this express is the first in the state (k_1, k_2, s, b). Then, the maximum lateness for the trains delivered to the destination station at passing from (k_1, k_2, s, b) to the next state is given by

$$L(k_1, k_2, s, b) = \begin{cases} \max\left\{p_1 + p_2 - d_s^{k_s}; 2p_s - d_{\bar{s}}^{k_{\bar{s}}}\right\} & \text{if } b = 2 \\ p_1 + p_2 - d_s^{k_s}, & \text{otherwise.} \end{cases}$$

Stated differently, at passage from type $(s, 2)$ to any express type admissible after it, delivered are one express and one non-express, in the other cases delivered is one express. Denote by $\Omega(k_1, k_2, s, b)$ the set of states that can occur after the state (k_1, k_2, s, b) according to Definition 6. With regard for Lemma 3 and except for the aforementioned initial states and states $(n_1, n_2, 2, 1)$ and $(n_1, n_2, 2, 2)$, as well as the states $(n_1, n_2, 1, 1)$ and $(n_1, n_2, 1, 2)$ for $p_1 < p_2 + \beta$, the Bellman equation is represented as follows:

$$F(k_1, k_2, s, b) = \min_{(k_1', k_2', s', b') \in \Omega(k_1, k_2, s, b)} \max\left\{L(k_1, k_2, s, b);\ F(k_1', k_2', s', b') + h((s, b), (s', b'))\right\}.$$

In the four last cases, a situation arises where the first express departs at the zero time instant, and the non-express making way for it is already at the siding line. To allow for the time required for the non-express to reach the siding line, add corresponding addends to the value of the objective function:

$$F(k_1, k_2, s, b) = \min_{(k_1', k_2', s', b') \in \Omega(k_1, k_2, s, b)} \max\left\{L(k_1, k_2, s, b);\ F(k_1', k_2', s', b') + h((s, b), (s', b'))\right\}$$
$$+ p_{\bar{s}} + \beta - p_s,$$

where (k_1, k_2, s, b) takes on the values $(n_1, n_2, 2, 1)$ and $(n_1, n_2, 2, 2)$, as well as $(n_1, n_2, 1, 1)$ and $(n_1, n_2, 1, 2)$ for $p_1 < p_2 + \beta$.

The minimal value of the state functions with $k_1 = n_1$ and $k_2 = n_2$ is the optimal value of the objective function of the initial problem $L_{\max}$:

$$L_{\max} = \min_{s \in \{1,2\}, b \in \{0,1,2\}} F(n_1, n_2, s, b). \tag{8}$$

The maximal number of computations of the values of the subproblem with different k_1 and k_2 is $O(n_1 n_2)$. Therefore, the algorithm complexity is $O(n^2)$ operations, where $n = n_1 + n_2$.

Remark. For the objective function $\sum w_i C_i$, the algorithm is constructed along the same lines. In the Bellman equations, the operation of selecting the maximal value max is replaced by summation. Also, according to Lemma 4 the shift of the beginning of time reading at calculating the next optimal value of the objective function of the subproblem is multiplied by the sum of all weights of the trains that were at the stations in the subproblem. The calculations begin with the states having the minimal number of undelivered trains:

$$F(1,0,1,0) = (p_1 + p_2) w_1^1;$$
$$F(0,1,2,0) = (p_1 + p_2) w_2^1;$$
$$F(1,1,2,2) = 2p_2 w_1^1 + (p_2 + p_1) w_2^1;$$
$$F(1,1,1,2) = 2p_1 w_2^1 + (p_2 + p_1) w_1^1.$$

With regard for Lemma 4, we establish that

$$F(k_1, k_2, s, b)$$
$$= \min_{(k_1', k_2', s', b') \in \Omega(k_1, k_2, s, b)} \left\{ \Sigma(k_1, k_2, s, b) + F(k_1', k_2', s', b') + h^{\Sigma}((k_1, k_2, s, b), (k_1', k_2', s', b')) \right\},$$

where

$$\Sigma(k_1, k_2, s, b) = \begin{cases} w_s^{k_s}(p_1 + p_2) + w_{\bar{s}}^{k_{\bar{s}}}(2p_s) & \text{if } b = 2 \\ w_s^{k_s}(p_1 + p_2), & \text{otherwise,} \end{cases}$$

$$h^{\Sigma}((k_1, k_2, s, b), (k_1', k_2', s', b')) = h((s, b), (s', b')) \sum_{i \in K_c',\, c \in \{1,\,2\}} w_c^i,$$

where $K_c' = \{1, \ldots, k_c'\}$ for $k_c' > 0$ and $K_c' = \emptyset$ for $k_c' = 0$, $c \in \{1, 2\}$.

Similar to the case of with $L_{\max}$, for the end states $(n_1, n_2, 2, 1)$ and $(n_1, n_2, 2, 2)$ and also for $p_1 < p_2 + \beta$ end states $(n_1, n_2, 1, 1)$ and $(n_1, n_2, 1, 2)$, the Bellman equation is given by

$$F(k_1, k_2, s, b)$$
$$= \min_{(k_1', k_2', s', b') \in \Omega(k_1, k_2, s, b)} \left\{ \Sigma(k_1, k_2, s, b) + F(k_1', k_2', s', b') + h^{\Sigma}((k_1, k_2, s, b), (k_1', k_2', s', b')) \right\}$$
$$+ (p_{\bar{s}} + \beta - p_s) \sum_{i \in N_c, c \in \{1,2\}} w_c^i.$$

The optimal value of the objective function $\sum w_i C_i$ is determined by analogy with (8):

$$\sum w_i C_i = \min_{s \in \{1,\,2\}, b \in \{0,\,1,\,2\}} F(n_1, n_2, s, b). \tag{9}$$

6. CONCLUSIONS

The present paper considered the problems of planning train traffic along a single railway track with a siding. On the basis of the method of dynamic programming for two objective functions of maximum lateness and weighted sum of arrival moments, an algorithm was developed with considering the problem properties such as the rules for passing between the states and inalterability of the order of train departures in the optimal schedule for the given transportation starting moment shifted by an arbitrary time interval. Complexity of the algorithm is $O(n^2)$ operations, where n is the number of trains at the stations.

It is planned to consider in what follows a more complicated model with more than one siding, siding lines of higher capacity, and different train speeds.

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