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Two-Station Single Track Scheduling Problem ⋆

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Abstract: Single track segments are common in various railway networks, in particular in various supply chains. For such a segment, connecting two stations, the trains form two groups, depending on what station is the initial station for the journey between these two stations. Within a group the trains differ by their cost functions. It is assumed that the single track is sufficiently long so several trains can travel in the same direction simultaneously. The paper presents polynomial-time algorithms for different versions of this two-station train scheduling problem with a single railway track. The considered models differ from each other by their objective functions.

Keywords: Operations Research, Transportation science, Supply chains and networks

1. THE CONSIDERED PROBLEM

The connection of two stations by a single railway track is common in various supply chains such as supply chains of minerals, for instance coal, and supply chains of agricultural products, for example sugar cane, as well as in the manufacturing environment. Due to their practical significance and challenging mathematical nature, the scheduling problems, where trains are using a single railway track, remained the subject of intensive research since the pioneer publications Frank (1966) and Szpigel (1973).

We investigate the problem where trains travel between two stations, Station 1 and Station 2, connected by a single track. Each train travels either from Station 1 to Station 2 (the set of all such trains will be denoted by N_1), or from Station 2 to Station 1 (the set of all such trains will be denoted by N_2). Let $N = N_1 \cup N_2$.

It is convenient to number all the trains departing from the same station, i.e. it is convenient to consider $N_1 = \{1, ..., n\}$ and $N_2 = \{1', ..., n'\}$. Observe, that $1', ..., n'$ are ordinary numbers assigned to the trains departing from Station 2. For example, $n' - 1'$ gives the number associated with one of the trains departing from Station 2, whereas $n + n'$ is the total number of trains.

All trains travel with the same constant speed. The journey between Station 1 and Station 2 takes $p > 0$ units of time. The transportation starts at time $t = 0$.

At any point in time, the distance between any two trains, simultaneously moving in the same direction, must be not less than the certain minimal safe distance. In order to ensure this restriction, the difference between any two departure times from the same station can not be less than some given β. In what follows, it is assumed that $\beta < p$. In other words, it is assumed that Station 1 and Station 2 are sufficiently far apart and several trains can travel simultaneously in the same direction. Observe that the difference between any two departure times from different stations can not be less than p.

A schedule σ specifies for each train $j \in N$ the departure time $S_j(\sigma)$ and the arrival time $C_j(\sigma)$, where

$$C_j(\sigma) = S_j(\sigma) + p.$$

Each train $j \in N$ has the associated nondecreasing cost function $\varphi_j(\cdot)$. The goal is to find a schedule which minimises the objective function

$$\varphi_1(C_1(\sigma)) \odot \ldots \odot \varphi_n(C_n(\sigma)) \odot \varphi_{1'}(C_{1'}(\sigma)) \odot \ldots \odot \varphi_{n'}(C_{n'}(\sigma)),$$

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where $\odot$ is some commutative and associative operation such that for any numbers a_1, a_2, b_1, b_2, satisfying $a_1 \leq a_2$ and $b_1 \leq b_2$,

$$a_1 \odot b_1 \leq a_2 \odot b_2. \tag{1}$$

For example, the operation $\odot$ can be addition. In this case for any $i \in N$ and $j \in N$

$$\varphi_i(C_i(\sigma)) \odot \varphi_j(C_j(\sigma)) = \varphi_i(C_i(\sigma)) + \varphi_j(C_j(\sigma)).$$

Another commonly used operation is maximum. In this case

$$\varphi_i(C_i(\sigma)) \odot \varphi_j(C_j(\sigma)) = \max\{\varphi_i(C_i(\sigma)), \varphi_j(C_j(\sigma))\}.$$

In particular, the objective function, where the operation is maximum and each $\varphi_i(x) = x$, is referred to in scheduling as the makespan and is denoted by C_{max}, i.e.

$$C_{max}(\sigma) = \max_{i \in N} C_i(\sigma).$$

Following the notation of the article Gafarov et al. (2015), the problem, considered in our paper, can be denoted by $STR2 \,||\, \odot\varphi_j$, where STR stands for "single track railway" and 2 indicates that two stations are considered.

Since the objective function is nondecreasing, in what follows, without loss of generality, it is assumed that each schedule σ should possess the following property: for any point in time t such that

$$0 \leq t \leq C_{max}(\sigma)$$

there exists at least one train $i \in N$ satisfying the condition

$$S_i(\sigma) \leq t \leq C_i(\sigma).$$

The rest of the paper is organised as follows. Section 2 discusses the contribution of the results, presented in this paper, to the existing knowledge on train scheduling and provides a brief literature survey. Section 3 shows that if the order in which the trains, constituting the set N_1, depart from Station 1 and the order in which the trains, comprising the set N_2, depart from Station 2 are known, then, for any operation $\odot$, an optimal schedule can be obtained in polynomial time by means of dynamic programming. This section also discusses some important cases of cost functions and operation $\odot$, where the computational complexity of the general optimisation procedure can be significantly reduced. Section 4 focuses on the maximum cost objective function frequently used in practice and theory.

2. EXISTING LITERATURE

Surveys of the railway planning processes, models and methods can be found in Lusby et al. (2011), Oliveira (2001) and Harrod (2012). The existing literature covers a broad variety of models, assumptions, and practical situations. Many publications stress the importance of the single track train scheduling from the practical as well as theoretical viewpoints. Indeed, the single track scheduling problems have attracted the considerable attention, and starting from Szpigel (1973), there exists a sturdy stream of publications.

In train scheduling, one of the parameters that vary from model to model is the speed of trains. For example, similar to our paper, Harbering et al. (2015) considers the situation where all trains travel with the same constant speed, whereas in Kraay et al. (1991) the speed can vary.

The optimisation methods also vary from publication to publication. Thus, similar to our paper, Harbering et al. (2015) uses dynamic programming. Other optimisation methods include integer programming Brannlund et al. (1998), branch and bound method Higgins et al. (1996), and heuristics Carey and Lockwood (1995) and Mu and Dessouky (2011). The computational complexity results can be found in Disser et al. (2015).

The existing publications on single track train scheduling consider a wide range of objective functions, including makespan, total tardiness, maximum lateness, total completion time, etc. Our paper demonstrates that many commonly used objective functions share certain properties that allow the development of uniform optimisation procedures.

The recent publication Gafarov et al. (2015) is closely related to our paper. Similar to our paper, Gafarov et al. (2015) is concerned with the two-station case and various objective functions (although not in such general form as in our paper). In contrast to our paper that considers the safe distance between the trains specified by β, Gafarov et al. (2015) assumes that the track is partitioned into several segments and trains are not allowed to travel simultaneously along each of these segments. These two assumptions are in some sense equivalent and, as far as Gafarov et al. (2015) is concerned, the main contribution of our paper is the development of more efficient optimisation algorithms for several objective functions, considered in Gafarov et al. (2015), as well as the presented optimisation procedure for the maximum cost problem with arbitrary nondecreasing cost functions.

Harbering et al. (2015) analyses the situation with more than two stations and presents a dynamic programming based pseudo-polynomial algorithm for the makespan minimisation problem. Harbering et al. (2015) considers the problem which is equivalent to a job shop scheduling with two routes. It is shown how to achieve a lower bound on the makespan when all operations of the job shop model have equal processing times.

3. DYNAMIC PROGRAMMING APPROACH

In this section we assume that the order in which the trains, constituting the set N_1, depart from Station 1 and the order in which the trains, comprising the set N_2, depart from Station 2 are known. These orders can be either specified by some properties of the objective function or determined by some factors outside the considered model, for example by trains priorities.

For example, consider the objective functions

$$L_{max}(\sigma) = \max_{i \in N}\{C_i(\sigma) - d_i\}, \tag{2}$$

and

$$\sum w_i C_i(\sigma) = \sum_{i \in N} w_i C_i(\sigma), \tag{3}$$

where, for each $i \in N$, d_i is the point in time by which it is desired to complete the journey of train i and w_i is a weight associated with train i. According to the following two lemmas, each of these objective functions specifies the order of departures for each station.

Lemma 1. There exists an optimal schedule for the problem $STR2\|L_{max}$, in which the trains from the set N_1 depart from Station 1 in a nondecreasing order of due dates d_i and the trains from the set N_2 depart from Station 2 in a nondecreasing order of due dates $d_{i'}$.

Lemma 2. There exists an optimal schedule for the problem $STR2\|\sum w_jC_j$, in which the trains from the set N_1 depart from Station 1 in a nonincreasing order of the weights w_i and the trains from the set N_2 depart from Station 2 in a nonincreasing order of the weights $w_{i'}$.

Let the trains at each station be numbered in the decreasing order of their departure times. Thus, for any $i \in N_s$ and $j \in N_s$, where $s \in \{1,2\}$, the inequality $i > j$ implies that, in any schedule σ, $S_i(\sigma) < S_j(\sigma)$. The following lemma describes the set of all possible departure times.

Lemma 3. In any schedule σ departure times of trains from Stations 1 and 2 belongs to the following set:

$$T = \{t \mid t = qp + k\beta,\ k \in \{0,1,\ldots,n+n'-2\},$$

$$q \in \{0,1,\ldots,2\min\{n,n'\}\}, k+q \le n+n'-1\}.$$

The number of elements in the set T is $O((n+n')^2)$.

For any $k_1 \in \{0,1,\ldots,n\}$ and $k_2' \in \{0',1',\ldots,n'\}$, such that either $k_1 \neq 0$, or $k_2' \neq 0'$, or both, and any $s \in \{1,2\}$, and $t \ge 0$, denote by $\mathcal{P}(k_1,k_2',s,t)$ the train scheduling problem that is obtained from the original single track train scheduling problem by the following assumptions:

- the set of trains that should travel from Station 1 to Station 2 is $\{1,\ldots,k_1\}$, i.e. ignore for a moment the existence of all trains in N_1 that do not belong to the set $\{1,\ldots,k_1\}$;
- the set of trains that should travel from Station 2 to Station 1 is $\{1',\ldots,k_2'\}$, i.e. ignore for a moment the existence of all trains in N_2 that do not belong to the set $\{1',\ldots,k_2'\}$;
- the original assumption that the first departure occurs at $t=0$ is replaced by the assumption that the time of the first departure is t;
- the first train must depart from the Station s, where s is part of the problem input.

Hence, if s in $\mathcal{P}(k_1,k_2',s,t)$ equals 1, then, in each schedule σ, $S_{k_1}(\sigma)=t$, and if $s=2$, then, in each schedule σ, $S_{k_2'}(\sigma)=t$.

Since $\mathcal{P}(k_1,k_2',s,t)$ is concerned with the set of trains

$$\{1,\ldots,k_1\}\cup\{1',\ldots,k_2'\},$$

its objective function contains the cost functions of these trains only. More specifically, in the case $k_1 \neq 0$ and $k_2' \neq 0'$ the objective function is

$$F(\sigma) = \bigodot_{i=1}^{k_1}\varphi_i(C_i(\sigma))\bigodot_{j=1'}^{k_2'}\varphi_j(C_j(\sigma)),$$

in the case $k_2' = 0'$,

$$F(\sigma) = \bigodot_{i=1}^{k_1}\varphi_i(C_i(\sigma))$$

and in the case $k_1 = 0$,

$$F(\sigma) = \bigodot_{j=1'}^{k_2'}\varphi_j(C_j(\sigma)).$$

It is convenient to denote the optimal value of the objective function for $\mathcal{P}(k_1,k_2',s,t)$ by $f(k_1,k_2',s,t)$. In other words,

$$f(k_1,k_2',s,t) = F(\sigma^*),$$

where σ^* is an optimal schedule for $\mathcal{P}(k_1,k_2',s,t)$. Using this notation, the optimal value of the objective function of the original single track train scheduling problem is

$$\min\{f(n,n',1,0), f(n,n',2,0)\}. \tag{4}$$

The values $f(n,n',1,0)$ and $f(n,n',2,0)$ can be obtained by solving a system of functional equations. In doing this, it is convenient to extend the operation $\odot$ by assuming that, for any number a,

$$a \odot \infty = \infty \odot a = \infty.$$

Furthermore, based on Lemma 3, it is convenient to amend the objective function for each $\mathcal{P}(k_1,k_2',s,t)$ by assuming that, for any schedule σ, in which at least one train has the departure time that is not in T, the value of the objective function is ∞.

The procedure of solving the system of functional equations begins by setting

$$f(1,0',1,t) = \begin{cases}\varphi_1(t+p), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases}$$

and

$$f(0,1',2,t) = \begin{cases}\varphi_{1'}(t+p), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases}$$

The subsequent computations are based on the property of the operation $\odot$, specified by the inequality (1). More specifically, for each $k_1 \in \{1,...,n-1\}$,

$$f(k_1+1,0',1,t) = \begin{cases} g(k_1+1,0',1,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

where

$$g(k_1+1,0',1,t) = \varphi_{k_1+1}(t+p)\odot f(k_1,0',1,t+\beta);$$

for each $k_2' \in \{1',...,n'-1'\}$,

$$f(0,k_2'+1',2,t) = \begin{cases} g(0,k_2'+1',2,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

where

$$g(0,k_2'+1',2,t) = \varphi_{k_2'+1'}(t+p)\odot f(0,k_2',2,t+\beta);$$

for each $k_2' \in \{1',...,n'\}$,

$$f(1,k_2',1,t) = \begin{cases} g(1,k_2',1,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

where

$$g(1,k_2',1,t) = \varphi_1(t+p)\odot f(0,k_2',2,t+p);$$

for each $k_1 \in \{1,...,n\}$,

$$f(k_1,1',2,t) = \begin{cases} g(k_1,1',2,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

where

$$g(k_1,1',2,t) = \varphi_{1'}(t+p)\odot f(k_1,0',1,t+p);$$

for each $k_1 \in \{1,...,n-1\}$ and each $k_2' \neq 0'$,

$$f(k_1+1,k_2',1,t) = \begin{cases} g(k_1+1,k_2',1,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

$$g(k_1+1,k_2',1,t)=\varphi_{k_1+1}(t+p)$$
$$\odot\min\{f(k_1,k_2',1,t+\beta),f(k_1,k_2',2,t+p)\};$$

for each $k_1 \neq 0$ and each $k_2' \in \{1',...,n'-1'\}$,

$$f(k_1,k_2'+1',2,t)=\begin{cases} g(k_1,k_2'+1',2,t), & \text{if } t\in T\\ \infty, & \text{if } t\notin T\end{cases},$$

$$g(k_1,k_2'+1',2,t)=\varphi_{k_2'+1'}(t+p)$$
$$\odot\min\{f(k_1,k_2',2,t+\beta),f(k_1,k_2',1,t+p)\}.$$

The number of operations needed to compute (4) is $O((n+n')^4)$, since it is necessary to compute value of f for each $t\in T$ and each pair of k_1 and k_2'.

3.1 Uniform objective functions

As above, this subsection assumes that the order in which trains leave Station 1 and the order in which trains leave Station 2 are known, but considers the situation when the set T is not required and therefore an optimal schedule can be constructed in $O((n+n')^2)$ operations.

Consider an arbitrary problem $\mathcal{P}(k_1,k_2',s,t)$, where $t\geq 0$, $k_1 \in \{0,1,\ldots,n\}$, and $k_2' \in \{0',1',\ldots,n'\}$, and either $k_1\neq 0$, or $k_2'\neq 0'$, or both. If

$$C_j(\eta)-C_j(\sigma)=t \quad \text{for all } j\in N \tag{5}$$

and η is a schedule for $\mathcal{P}(k_1,k_2',s,t)$, then σ is a schedule for $\mathcal{P}(k_1,k_2',s,0)$ and vice versa. For any schedule σ for $\mathcal{P}(k_1,k_2',s,0)$, the schedule η, satisfying (5), will be denoted by σ_t.

For an arbitrary problem $\mathcal{P}(k_1,k_2',s,t)$, let $F_t^{k_1k_2'}$ be the objective function of $\mathcal{P}(k_1,k_2',s,t)$. Observe that each $F_t^{k_1k_2'}$ uses the same cost functions as the objective function of the original train scheduling problem and the same operation $\odot$. This original objective function is uniform if there exists a function $G(k_1,k_2',t)$ such that, for any $\mathcal{P}(k_1,k_2',s,0)$, any schedule σ for $\mathcal{P}(k_1,k_2',s,0)$, and any $t>0$,

$$F_t^{k_1k_2'}(\sigma_t)=F_0^{k_1k_2'}(\sigma)+G(k_1,k_2',t).$$

It is easy to see that this property implies that if σ^* is an optimal schedule for $\mathcal{P}(k_1,k_2',s,0)$, then, for any $t>0$, σ_t^* is an optimal schedule for $\mathcal{P}(k_1,k_2',s,t)$. In other words, an optimal schedule for $\mathcal{P}(k_1,k_2',s,t)$ is obtained by "shifting" an optimal schedule for $\mathcal{P}(k_1,k_2',s,0)$ by t. Consequently, in computing (4), it is necessary to compute $f(k_1,k_2',s,0)$ only. Since the last argument in $f(k_1,k_2',s,0)$ never changes, it can by omitted.

More specifically, the computation starts with setting

$$f(1,0',1)=\varphi_1(p)$$

and

$$f(0,1',2)=\varphi_{1'}(p).$$

Then, for each $k_1\in\{1,...,n-1\}$,

$$f(k_1+1,0',1)=\varphi_{k_1+1}(p)\odot[f(k_1,0',1)+G(k_1,0',\beta)];$$

for each $k_2'\in\{1',...,n'-1'\}$,

$$f(0,k_2'+1',2)=\varphi_{k_2'+1'}(p)\odot[f(0,k_2',2)+G(0,k_2',\beta)];$$

for each $k_2'\in\{1',...,n'\}$,

$$f(1,k_2',1)=\varphi_1(p)\odot[f(0,k_2',2)+G(0,k_2',p)];$$

for each $k_1\in\{1,...,n\}$,

$$f(k_1,1',2)=\varphi_{1'}(p)\odot[f(k_1,0',1)+G(k_1,0',p)];$$

for each $k_1\in\{1,...,n-1\}$ and each $k_2'\neq 0'$,

$$f(k_1+1,k_2',1)=\varphi_{k_1+1}(p)$$
$$\odot\min\{f(k_1,k_2',1)+G(k_1,k_2',\beta),$$
$$f(k_1,k_2',2)+G(k_1,k_2',p)\};$$

for each $k_1\neq 0$ and each $k_2'\in\{1',...,n'-1'\}$,

$$f(k_1,k_2'+1',2)=\varphi_{k_2'+1'}(p)$$
$$\odot\min\{f(k_1,k_2',1)+G(k_1,k_2',p),$$
$$f(k_1,k_2',2)+G(k_1,k_2',\beta)\}.$$

The important examples of uniform objective functions are the objective functions (2) and (3). Indeed, for (2),

$$G(k_1,k_2',t)=t,$$

whereas for (3),

$$G(k_1,k_2',t)=\sum_{i=1}^{k_1}w_it+\sum_{j=1'}^{k_2'}w_jt.$$

As has been shown above, both, (2) and (3), specify the order in which trains leave Station 1 and the order in which trains leave Station 2.

4. MINIMIZATION OF MAXIMUM COST

This section is concerned with the objective function

$$F_{max}(\sigma)=\max_{i\in N}\varphi_i(C_i(\sigma)), \tag{6}$$

where all $\varphi_i(\cdot)$ are nondecreasing cost functions. The algorithm below is an iterative optimisation procedure based on the general optimisation scheme, presented in Zinder and Shkurba (1985), and the dynamic programming based algorithm for $STR2\|L_{max}$, presented in Subsection 3.1. The optimisation scheme, described in Zinder and Shkurba (1985), is applicable to various scheduling problems with the objective function (6), satisfying two conditions:

(C1) for each instance of the scheduling problem, the set of all possible completion times is a subset of a known set which cardinality is bounded above by some polynomial in a parameter specifying the size of this instance (this polynomial remains the same for all instances);

(C2) there exists a polynomial-time algorithm which solves the scheduling problem that is obtained from the considered scheduling problem by replacing the original objective function (6) by the maximum lateness (2).

In the considered train scheduling problem, let

$$T'=\{\tau_1,\tau_2,\ldots,\tau_r\}$$

be a set that contains the set of all possible arrival times (completion times), where

$$\tau_1<\ldots<\tau_r.$$

The elements of T' can be obtained from the set T, specified in Lemma 3, by adding p to each element of T. Hence, the cardinality of T' is $O((n+n')^2)$ and this set T' satisfies the condition $(C1)$.

According to Zinder and Shkurba (1985), at each iteration, a lower bound on (6) is computed together with the due

Algorithm 1 Solution method for the train scheduling problem $STR2\|F_{max}$

```
1: V := max_{i∈N} φ_i(p) (lower bound)
2: for i := 1 to n + n' do
3:   if φ_i(τ_r) ≤ V then
4:     d_i := τ_r
5:   else
6:     choose τ_k so that φ_i(τ_k) ≤ V < φ_i(τ_{k+1})
7:     d_i := τ_k
8:   end if
9: end for
10: construct schedule σ by solving STR2||L_max
11: L := L_max(σ)
12: if L > 0 then
13:   V := min_{i∈{j: j∈N, (d_j+L)∈T'}} φ_i(d_i + L) (lower bound)
14:   go to 2
15: else
16:   return σ is an optimal value
17: end if
```

dates that guarantee that the value of (6) will not exceed this lower bound.

In Algorithm 1, the initial value of the lower bound V is calculated according to the line 1. Indeed, the arrival time of train i cannot be less than p. Hence, the cost for train i cannot be less than $\varphi_i(p)$. Therefore, the maximum cost cannot be less than $\max_{i\in N} \varphi_i(p)$.

At each iteration, according to lines 3 and 4 train i is assigned the largest element of the set T' as a due date if this assignment does not violate the lower bound V. If this assignment violates V, then according to the lines 6 and 7, the due date is chosen as the largest element in T' that does not violate V.

After all due dates have been assigned, the maximum lateness problem is solved. If for the resultant schedule σ the maximum lateness $L_{max}(\sigma) \leq 0$, then for this schedule, $F_{max}(\sigma)$ equals the lower bound and therefore σ is optimal for (6). If $L_{max}(\sigma) > 0$, then a new lower bound is computed according to line 13. The new value of V is a new lower bound because for any schedule at least for one train i the delay will be not less than L. The new lower bound, computed according to line 13, is greater than the previous one and a new iteration begins.

As the algorithm for the maximum lateness problem $STR2\|L_{max}$ has complexity $O((n+n')^2)$, the complexity of the iterative algorithm is $O((n+n')^5 \log(n+n'))$. As has been mentioned above, the due dates in the maximum lateness problem specify the order in which trains depart from Station 1 and the order in which trains depart from Station 2. Since the set of due dates changes from iteration to iteration, these orders also change from iteration to iteration.

5. CONCLUSION

The paper presents two polynomial-time optimisation procedures for the train scheduling problem where two stations are connected by a single railway track. The first procedure is based on dynamic programming and the assumption that for each station the order in which trains depart from this station is know. The procedure is applicable to a family of objective functions that includes a number of objective functions frequently used in theory and practice. For some of these objective functions the order of departures from each station can be deduced from the analysis of the structure of an optimal schedule. For these objective functions, the presented optimisation procedure is significantly more efficient than the previously known algorithms. For example, the minimisation of $\sum w_i C_i$ requires $O((n+n')^2)$ operations, whereas the previously known algorithm required $O((n+n')^6)$ operations Gafarov et al. (2015).

The second optimisation procedure is an iterative algorithm developed for the minimisation of maximum cost. This algorithm uses the first optimisation procedure as a subroutine for minimising at each iteration the maximum lateness for the set of due dates that changes at each iteration. Since for the maximum lateness there exists an optimal schedules where the trains depart from their respective stations in a nondecreasing order of there due dates, these orders change from iteration to iteration. The second polynomial-time optimisation procedure is applicable for any nondecreasing cost functions.

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