# A Measure of Functions' Asymptotic Growth and the Complexity Classification of Computer Algorithms

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## ABSTRACT
In this paper, a special angular measure of functions' asymptotic growth is offered, which allows one to distinguish between sub-polynomial, polynomial, sub-exponential, exponential and super-exponential functions. On the basis of the measure, an algorithm computational complexity classification is introduced oriented to application in theoretical and practical comparative analysis of computer algorithms.

**Keywords:** *Algorithms, algorithm complexity, functions' asymptotic growth measure, complexity classification of the algorithms*

## 1. INTRODUCTION

In the development of algorithms and mathematics for software systems, the analysis is concerned with problem-solving algorithms and their software implementation. The impressive growth of contemporary computers' performance, apparently, should not weaken the requirements for algorithmic support, including the requirements for implementation efficiency. This is due to the fact that a number of actual computational tasks belong either to the $NP$-complete or to the $NP$-hard class, which have exact solution algorithms with a super-polynomial complexity [1, 2, 3]. A significant increase of dimensionality is another characteristic of present-day computational tasks; problems which are solved using the finite element method, and especially inverse problems, could be considered examples [4]. Thereby, the problem of selecting rational algorithms' appears to be of major importance. In order to solve such a problem efficiently, classifications of algorithms may be introduced, basing on complexity functions' estimates.

In the algorithmic theory, the traditional classification is established on the results obtained by Cobham (1964) [5], Cook (1971) [6] and Levin (1973) [7]. These studies are believed to have laid foundation to the computational complexity theory, which focuses on, and had a number of significant results related to, problem classes [8]. The theory considers a few particular problem classes; the majority of solvable problems lie within $P$, a class that encompasses problems with polynomial-complexity solution algorithms. As for the $NP$-complete problems, the majority of their precise solution algorithms known to date have either exponential ($EXP$ class) or super-polynomial complexity [1, 2, 8, 9].

Unfortunately, for the analysis of algorithms' resource efficiency and for the complexity classification, the usage of classical complexity theory seems not to be quite correct. This is due to the fact that the complexity theory operates with problem classes, not algorithm classes; the majority of problem class definitions do not explicitly denote any complexity bounds, with an exception for $P$ and $EXP$ classes [1, 8].

## 2. STATEMENT OF PROBLEM

For theoretical research of computer algorithms' computational complexity, development of a well-posed complexity function classification appears to be of a particular interest. The classification could be made on the basis of a detailed delineation of asymptotic estimates for algorithms' complexity functions, with preservation of traditionally distinguished polynomial and exponential functions. Thereby, it comes to a mathematical task of separating polynomials from exponentials within a unified measure. Such a measure should also indicate a set of functions that separate polynomials and exponentials, as well as sets of sub-polynomial and super-exponential functions. The separation method may serve as a basis for correct computational complexity classification of computer algorithms.

## 3. TERMINOLOGY AND NOTATION

Primarily abiding by [10] and the article of E. L. Post [11], let us use the following notation and terminology relative to algorithms' computational complexity:

$Z$ — The abstract denotation of the problem in Post's terminology;

$A$ — The abstract denotation of a particular algorithm that solves $Z$ (a finite 1-process that solves the general problem [11]);

$D$ — An input of $A$ — a finite set of fixed-length binary words, such that it determines a specific problem for the general problem $Z$;

$D_A = \{D\}$ — The set of allowable inputs for the algorithm $A$ solving the problem $Z$;

$\lambda(D)$ — The length of algorithm's input:

$$D \underset{\lambda}{\rightarrow} N,$$

An integer-valued function, commonly defined as the cardinality of $D: \lambda(D) = |D| = n$; in particular cases (for example, for the matrix multiplication problem) — a certain function of the cardinality: $\lambda(D) = g(|D|)$;

$f_A(D)$ — the computational complexity of $A$ on input $D$, an integer-valued function that indicates the amount of basic operations specified by the algorithm $A$ on the input $D$ in the accepted computational model;
$D_n$ — the set of inputs of length $n$ for $A$:

$$D_n = \{D \mid D \in D_A, \lambda(D) = n\};$$

$f_A^{\wedge}(n)$ — the worst-case computational complexity of the algorithm on inputs of length $n$, that is, the maximum value of $f_A(D)$ on the set $D_n$;

$f(n)$ — the complexity of the algorithm — the function that appears (either in $O$- or $\Theta$-notation) in the asymptotic estimate for $f_A^{\wedge}(n)$ — the worst-case computational complexity function of the algorithm:

$$f_A^{\wedge}(n) = O(f(n)), \; or \; f_A^{\wedge}(n) = \Theta(f(n)).$$

For further discussion, the argument will be assumed to be continuous, that is, $f(\cdot) = f(x)$, all the required values being obtained at integer points $x = n$.

## 4. FUNCTIONS SEPARATING POLYNOMIALS AND EXPONENTIALS

From the set of functions separating polynomials from exponentials, let us select the exponential-logarithmic function as illustrative: $g(x) = (\ln x)^{\ln x}$.

**Statement 1:**

The exponential-logarithmic function $g(x) = (\ln x)^{\ln x}$ is a separating function for polynomials and exponentials.

**Proof:**

The statement is equivalent to an assertion that $g(x)$ satisfies the following two implications when $x \to \infty$:

$$\text{if } f(x) = x^k, \; k > 0, \text{ then } f(x) = o(g(x)), \quad (1)$$
$$\text{if } f(x) = e^{\lambda x}, \; \lambda > 0, \text{ then } g(x) = o(f(x)). \quad (2)$$

To prove these relations, let us apply the logarithmic limit lemma [12]: if

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to\infty} g(x) = \infty,$$

then the following implication holds true:

$$\text{If } \lim_{x\to\infty} \frac{\ln f(x)}{\ln g(x)} = 0, \text{ that } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0, \text{ that is,}$$

$$f(x) = o(g(x)).$$

Under this lemma, the validity of the implication (1) can be proven:

$$\lim_{x\to\infty} \frac{\ln(x^k)}{\ln\left((\ln x)^{\ln x}\right)} = \lim_{x\to\infty} \frac{k \cdot \ln x}{\ln x \cdot \ln(\ln x)} = \lim_{x\to\infty} \frac{k}{\ln(\ln x)} = 0,$$

thereby, $x^k = o\left((\ln x)^{\ln x}\right)$ if $k > 0$. The validity of implication (2) is proven analogically:

$$\lim_{x\to\infty} \frac{\ln\left((\ln x)^{\ln x}\right)}{\ln\left(e^{\lambda x}\right)} = \lim_{x\to\infty} \frac{\ln x \cdot \ln(\ln x)}{\lambda x} = 0,$$

thereby, $(\ln x)^{\ln x} = o\left(e^{\lambda x}\right)$ when $\lambda > 0$.
End of proof.

## 5. PRELIMINARY LEMMAS AND COORDINATE SYSTEM TRANSFORMATION

For the problem of separating polynomials from exponents, a solution is offered based on a unified measure, which distinguishes additionally the sets of separating, sub-polynomial, and super-exponential functions.

Let $f = f(x)$ be a monotonously increasing function such that $\lim_{x\to\infty} f(x) = \infty$.

Let us introduce an operator $H(f(x)) = h(x)$ which associates $f(x)$ with a function $h(x)$ by the following rule:

$$H(f(x)) = h(x) = \ln(f(x)) + \frac{\ln(f(x))}{\ln(f(x)) + \ln x} \cdot x. \quad (3)$$

The function $h(x) = H(f(x))$ possesses the following property — the limit of the derivative of $h(x)$ is a constant for both polynomials and exponentials:

$$\lim_{x\to\infty} h'(x) = C, \; C > 0, \quad (4)$$

The property is established by two following lemmas:

**Lemma 1:**

Let $f(x) = e^{\lambda x}(1 + \alpha(x))$, where

$\lambda > 0, \alpha(x) = o(1), \alpha'(x) = o(1)$, when $x \to \infty$;

then

$$\lim_{x\to\infty} h'(x) = \lambda + 1.$$

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**Proof:**

The derivative $h'(x)$ of $f(x)$ can be calculated using the definition (3):

$$h'(x)=\frac{f'}{f}+\frac{\ln f}{\ln f+\ \ln x}\cdot 1\ + \\ +\ x\cdot\frac{\frac{f'}{f}\cdot(\ln f+\ln x)\ -\ \ln f\cdot\left(\frac{f'}{f}+\frac{1}{x}\right)}{(\ln f+\ \ln x)^2}\quad = \qquad (5) \\ =\ \frac{f'}{f}+\frac{\ln f}{\ln f+\ \ln x}+x\cdot\frac{\frac{f'}{f}\cdot\ln x\ -\ \frac{1}{x}\cdot\ln f}{(\ln f+\ \ln x)^2}.$$

For the obtained derivative, let us compute the limits of the summands for $f(x)=\mathrm{e}^{\lambda x}(1+\alpha(x))$, in respect that $\lambda>0,\ \alpha(x)=o(1),\ \alpha'(x)=o(1)$:

$$\lim_{x\to\infty}\frac{f'}{f}=\lim_{x\to\infty}\frac{\lambda e^{\lambda x}(1+\alpha(x))+e^{\lambda x}\alpha'(x)}{e^{\lambda x}(1+\alpha(x))}= \\ =\lim_{x\to\infty}\frac{\lambda\cdot(1+\alpha(x))+\alpha'(x)}{(1+\alpha(x))}=\lambda;$$

$$\lim_{x\to\infty}\frac{\ln f}{\ln f+\ln x}=\lim_{x\to\infty}\frac{\lambda x\ +\ \ln(1+\alpha(x))}{\lambda x\ +\ \ln(1+\alpha(x))\ +\ \ln x}\ =\ 1;$$

$$\lim_{x\to\infty}x\cdot\frac{\frac{f'}{f}\cdot\ln x-\frac{1}{x}\cdot\ln f}{(\ln f+\ln x)^2}= \\ =\lim_{x\to\infty}x\cdot\frac{\frac{\lambda e^{\lambda x}(1+\alpha)+e^{\lambda x}\alpha'}{e^{\lambda x}(1+\alpha)}\cdot\ln x-\frac{1}{x}\cdot(\lambda+\ln(1+\alpha))}{(\lambda+\ln\ (1+\alpha)+\ \ln x)^2}\ =$$

$$=\lim_{x\to\infty}\frac{\frac{\lambda(1+\alpha)+\alpha'}{(1+\alpha)}\cdot\frac{\ln x}{x}\ -\ \frac{\lambda+\frac{\ln\ (1+\alpha)}{x}}{x}}{\left(\lambda+\ln\ (1+\alpha)+\ \frac{\ln x}{x}\right)^2}\ =\ 0,$$

thereby, $\lim\limits_{x\to\infty}h'(x)\ =\ \lambda+1$.

End of proof.

**Lemma 2:**

Let $f(x)=x^k(1+\alpha(x))$,

where

$$\alpha(x)=o(1),\ \alpha'(x)=o(1),\ x\alpha'(x)=O(1),$$

When $x\to\infty$;

then

$$\lim_{x\to\infty}h'(x)\ =\ \frac{k}{k+1}.$$

**Proof:**

Let us use formula (5) from lemma 1 for $h'(x)$ and calculate the limits of the summands when $x\to\infty$ for

$$f(x)\ =\ x^k(1+\alpha(x)):$$

$$\lim_{x\to\infty}\frac{f'}{f}=\lim_{x\to\infty}\frac{kx^{k-1}(1+\alpha(x))+x^k\alpha'(x)}{x^k(1+\alpha(x))}= \\ =\lim_{x\to\infty}\left[\frac{k}{x}+\frac{\alpha'(x)}{(1+\alpha(x))}\right]=0;$$

$$\lim_{x\to\infty}\frac{\ln f}{\ln f+\ln x}=\lim_{x\to\infty}\frac{k\ln x+\ln(1+\alpha(x))}{k\ln x+\ln(1+\alpha(x))+\ln x}= \\ =\lim_{x\to\infty}\frac{k+\frac{\ln(1+\alpha(x))}{\ln x}}{k+\frac{\ln(1+\alpha(x))}{\ln x}+1}=\frac{k}{k+1};$$

$$\lim_{x\to\infty}\frac{\frac{f'}{f}\ln x-\frac{1}{x}\ln f}{(\ln f+\ln x)^2}= \\ =\lim_{x\to\infty}\frac{\frac{kx^{k-1}(1+\alpha)+x^k\alpha'}{x^k(1+\alpha)}\ln x-\frac{1}{x}(k\ln x+\ln(1+\alpha(x)))}{(k\ln x+\ln(1+\alpha(x))+\ln x)^2}= \\ =\lim_{x\to\infty}\frac{k+\frac{\alpha'\ \ln\ x}{(1+\alpha)}\cdot x\ -\ (k\ \ln\ x+\ln\ (1+\alpha))}{\ln^2 x\left[k\ +\frac{\ln\ (1+\alpha)}{\ln\ x}+\ 1\right]^2}\ = \\ =\lim_{x\to\infty}\frac{1}{(k+1)^2}\left[\frac{k}{\ln^2 x}+\frac{\alpha'\cdot x}{(1+\alpha)\cdot\ln\ x}-\frac{k}{\ln\ x}-\frac{\ln\ (1+\alpha)}{\ln^2 x}\right]= \\ =0,$$

thereby,

$$\lim_{x\to\infty}h'(x)=k/(k+1).$$

End of proof.

Due to the fact that the limit of the derivative of function $h(x)=H(f(x))$ is a constant for both polynomials and exponentials (according to lemmas 1 and 2), the following lemma can be proved, introducing a coordinate system transformation.

**Lemma 3:**

Let $h(x)$ be a function such that $\lim\limits_{x\to\infty}h'(x)=C>0$.

Let us introduce a parametric function $z(s)$, created on the basis of $h(x)$ according to the following rule:

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$$z(s)=\begin{cases} s = arctg\left(\dfrac{1}{x}\right), \\ z = arctg\left(\dfrac{1}{h(x)}\right), \end{cases} \qquad (6)$$

then

$$\lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = \frac{1}{C}. \qquad (7)$$

**Proof:**

By the conditions of the lemma, $x \to \infty$ implies $h(x) \to \infty$; then, by the definition of function $z(s)$ — formula (6), $s \to 0$, $z \to 0$. In these circumstances, the definition of $z(s)$ can be extended as follows: let $z = 0$ when $s = 0$ $(x \to \infty)$. Then

$$\frac{dz}{ds} = \frac{\dfrac{dz}{dx}}{\dfrac{ds}{dx}} = \frac{\dfrac{1}{1+\dfrac{1}{h^2}}\left(-\dfrac{h'}{h^2}\right)}{\dfrac{1}{1+\dfrac{1}{x^2}}\cdot\left(-\dfrac{1}{x^2}\right)} = \frac{\left(x^2+1\right)\cdot h'}{h^2+1}.$$

Consider

$$\lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = \lim_{x\to\infty} \frac{\left(x^2+1\right)\cdot h'(x)}{h(x)^2+1} = \lim_{x\to\infty} \frac{x^2 \cdot h'(x)}{h(x)^2};$$

observing that

$$\lim_{x\to\infty} \frac{h(x)}{x} = \lim_{x\to\infty} \frac{h'(x)}{1} = C,$$

the result is

$$\lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = \lim_{x\to\infty} \frac{x^2 \cdot h'(x)}{h(x)^2} = \lim_{x\to\infty} h'(x)\cdot\frac{x^2}{h(x)^2} =$$

$$= C \cdot \frac{1}{C^2} = \frac{1}{C}.$$

End of proof.

## 6. GRAPHICAL INTERPRETATION OF THE COORDINATE SYSTEM $(z, s)$

The result obtained of lemmas 1, 2 and 3, can be interpreted graphically in the following way: in the $(z, s)$ coordinate system polynomials and exponents are mapped to $z(s)$ functions, which have different slopes as $x \to \infty$, that is, in the transformed coordinate system, different slopes at the point $(z = 0, s = 0)$ when $s \to 0$. Two examples of $z(s)$ functions obtained from the formula (6) for $f(x) = x^2$ и $f(x) = e^x$ are shown in Figure 1.

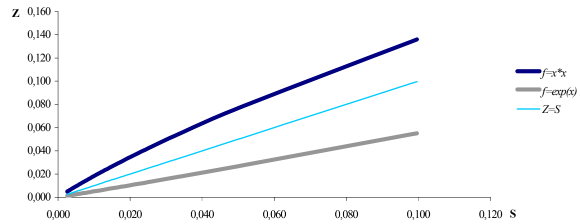


**Fig 1:** $z(s)$ function for a polynomial ( $f(x) = x^2$ ) and an exponent ( $f(x) = e^x$ ).

## 7. ANGULAR MEASURE OF FUNCTIONS' ASYMPTOTIC GROWTH

Diverse $z(s)$ slopes for polynomials and exponentials in the asymptotic behavior when $x \to \infty$, that is, when $s \to 0$ in the transformed coordinate systems, allows a new measure to be introduced for asymptotic growth of functions. Slope difference has determined a proposal for its name — the angular measure of functions' asymptotic growth.

Lemmas 1, 2 and 3 serve as a basis for the following theorem.

**Theorem 1:**

(of the angular measure of functions' asymptotic growth).

Let $f = f(x)$ be a monotonously increasing function, such that $\lim\limits_{x\to\infty} f(x) = \infty$. We define a measure $\gamma(f(x))$ for the asymptotic (at infinity) growth of the function:

$$\gamma(f(x)) = \pi - 2\cdot arctg(R), \text{ where } R = \lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds},$$

Parametrically defined function $z(s)$ is specified in (6), where $h(x)$ is obtained by applying operator $H$ to function $f(x)$:

$$H(f(x)) = h(x) = \ln(f(x)) + \frac{\ln(f(x))}{\ln(f(x)) + \ln x}\cdot x;$$

Thereby, if

1) $f(x) = e^{\lambda x}(1+\alpha(x))$, where $\lambda > 0, \alpha(x) = o(1), \alpha'(x) = o(1)$, and $x \to \infty$, then
$$\pi/2 < \gamma(f(x)) < \pi;$$

2) $f(x) = x^k(1+\alpha(x))$, where $\alpha(x) = o(1), \alpha'(x) = o(1), x\alpha'(x) = O(1)$, and $x \to \infty$, then
$$0 < \gamma(f(x)) < \pi/2;$$

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3) $f(x) = \ln x^{\ln x}$, then

$$\gamma(f(x)) = \pi/2.$$

**Proof:**

The first assertion of the theorem can be proved as follows. If $f(x) = e^{\lambda x}(1+\alpha(x))$, then the limit of the derivative of $h(x)$, which is defined upon $f(x)$, $x \to \infty$, is equal to $\lim\limits_{x\to\infty} h'(x) = \lambda + 1$, where $\lambda > 0$, by Lemma 1, herewith

$$R = \lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = \frac{1}{1+\lambda}$$

by Lemma 3; thereby, the values of $\lambda$ that are possible for the $EXP$ class, $0 < \lambda < \infty$, lead to the inequality $\pi/4 > arctg(R) > 0$; this determines the range of corresponding measure values: $\pi/2 < \gamma(f(x)) < \pi$, by the definition of $\gamma(f(x))$. The first assertion of the theorem has been proved.

Let us prove the second assertion of the theorem. If $f(x) = x^k(1+\alpha(x))$, then the limit of the derivative of $h(x)$, which is defined upon $f(x)$, $x \to \infty$, is equal to $\lim\limits_{x\to\infty} h'(x) = \dfrac{k}{k+1}$, where $k > 0$, by Lemma 2, herewith

$$R = \lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = \frac{k+1}{k}.$$

By Lemma 3; thereby, the values of $k$ that are possible for the $P$ class, $0 < k < \infty$, lead to the inequality $\pi/2 > arctg(R) > \pi/4$; this determines the range of corresponding measure values: $0 < \gamma(f(x)) < \pi/2$, by the definition of $\gamma(f(x))$. This proves the second assertion of the theorem.

Let us prove the third assertion of the theorem. Let us consider $h(x)$ defined upon the exponential-logarithmic function $f(x) = (\ln x)^{\ln x}$:

$$h(x) = \ln x \ \ln\ln x + x\frac{\ln x \ \ln\ln x}{\ln x \ \ln\ln x + \ln x} = \\ = \ln x \ \ln\ln x + x\frac{\ln\ln x}{\ln\ln x + 1};$$

The limit of the derivative of $h(x)$, as $x \to \infty$, can be calculated as follows:

$$\lim_{x\to\infty} h'(x) = \lim_{x\to\infty}\left(\frac{\ln\ln x}{x} + \ln x \frac{1}{\ln x}\frac{1}{x}\right) + \\ + \lim_{x\to\infty}\left(\frac{\ln\ln x}{\ln\ln x + 1}\cdot 1 + x\frac{\frac{1}{\ln x}\frac{1}{x}(\ln\ln x + 1) - \ln\ln x\frac{1}{\ln x}\frac{1}{x}}{(\ln\ln x + 1)^2}\right) = 1,$$

then, by Lemma 3,

$$R = \lim_{\substack{x\to\infty \\ (s\to 0)}} \frac{dz}{ds} = 1,$$

and, consequently, $\operatorname{arctg}(R) = \pi/4$, so that

$$\gamma\left((\ln x)^{\ln x}\right) = \pi/2.$$

End of proof.

## 8. PROPERTIES OF THE ANGULAR MEASURES OF FUNCTIONS' ASYMPTOTIC GROWTH

The proposed measure of functions' asymptotic growth possesses a number of properties that allow it to be used for constructing an algorithm complexity classification. Foremost, let us define the following five functional sets on the basis of the introduced measure $\gamma(f(x))$, assuming that $\lim\limits_{x\to\infty} f(x) = \infty$:

1) The set $FZ$ is defined as follows:

$$FZ = \left\{ f(x) \mid f(x) \prec x^k, \forall k > 0 \right\},$$

forming a set of sub-polynomial functions. For each function $f(x)$ in the set $FZ$ the value of $R$ as determined by Lemma 3, equals $+\infty$, so that the measure $\forall f(x) \in FZ$ $\gamma(f(x)) = \pi - 2\cdot\operatorname{arctg}(R) = 0$. In particular, $\gamma(\ln(x)) = 0$.

2) The set $FP$ may be defined as follows:

$$FP = \left\{ f(x) \mid \exists k > 0 : f(x) = \Theta(x^k) \right\},$$

forming a set of polynomial functions. This definition is based on Lemma 2; it can be proved, though, that the proposed measure still holds the same value for a broader class of functions $f(x) = \Theta(x^k)\cdot g(x)$, where $g(x) \in FZ$. Thereby, the set $FP$ may be defined otherwise. On the basis of the initially defined set $F_k$:

$$F_k = \left\{ f(x) \mid x^{k-\varepsilon} \prec f(x) \prec x^{k+\varepsilon}, k > 0, \varepsilon \to +0, x \to +\infty \right\}$$

we define the generalized polynomial set $FP$ on its basis:

$$FP = \left\{ f(x) \mid \exists k > 0 : f(x) \in F_k \right\}.$$

The function $f(x)$ in the definition of $FP$ gives the value of $R = (k+1)/k$, $k > 0$, by Lemmas 2 and 3, so that the measure $\gamma(f(x)) = \pi - 2\cdot\operatorname{arctg}((k+1)/k)$, thereby, $0 < \gamma(f(x)) < \pi/2$. The plot of the measure for polynomials is shown in Figure 2.

3) The set $FL$ is defined as follows:

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$$FL = \{ f(x) \mid x^k \prec f(x) \prec e^{\lambda x}, \forall k > 0, \forall \lambda > 0 \},$$

forming a set of sub-exponential functions. For any function $f(x)$ belonging to $FL$ the value of $R$, as defined by Lemma 3, equals 1, so that the measure $\gamma(f(x)) = \pi - 2 \cdot \mathrm{arctg}(1)$. In particular, $\gamma(\ln x^{\ln x}) = \pi/2$.

4) The set $FE$ may be defined as follows:

$$FE = \{ f(x) \mid \exists \lambda > 0 : f(x) = \Theta(e^{\lambda x}) \},$$

forming a set of exponential functions. This definition is based on Lemma 3, but one can prove, that the proposed measure still holds the same value for a broader class of functions of the form $f(x) = \Theta(e^{\lambda x}) \cdot g(x)$, where $g(x)$ belongs to one of the sets $FZ, FP, FL$. Thereby, the set $FE$ may be defined otherwise. On the basis of the initially defined set $F_\lambda$:

$$F_\lambda = \left\{ \begin{array}{l} f(x) \mid e^{(\lambda - \varepsilon) x} \prec f(x) \prec e^{(\lambda + \varepsilon) x}, \\ \lambda > 0, \varepsilon \to +0, x \to +\infty \end{array} \right\},$$

we define the generalized exponential set $FE$:

$$FE = \{ f(x) \mid \exists \lambda > 0 : f(x) \in F_\lambda \}.$$

For each function $f(x)$ that belongs to $FE$ the value of $R = 1/(1+\lambda), \lambda > 0$, by Lemmas 1 and 3, so that the measure

$$\gamma(f(x)) = \pi - 2 \cdot \mathrm{arctg}(1/(1+\lambda)),$$

thereby $\pi/2 < \gamma(f(x)) < \pi$. The plot of the measure for exponentials is presented in Figure 3.

5) The set $FF$ is defined as follows:

$$FF = \{ f(x) \mid e^{\lambda x} \prec f(x), \forall \lambda > 0 \},$$

forming a set of super-exponential functions. For function $f(x)$ in the definition of $FF$ the value of $R$, as defined by Lemma 3, equals zero, so that the measure $\gamma(f(x)) = \pi - 2 \cdot \mathrm{arctg}(R) = \pi$. In particular, $\gamma(x^x) = \pi$.

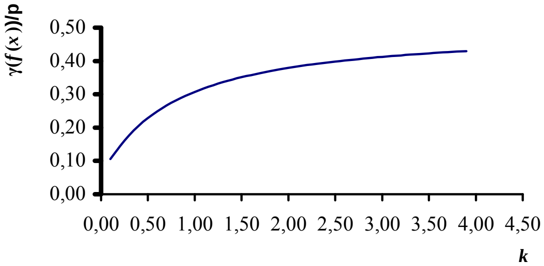


**Fig 2:** The plot of the measure $\gamma(f(x))$ for polynomials $f(x) = \Theta(x^k)$.

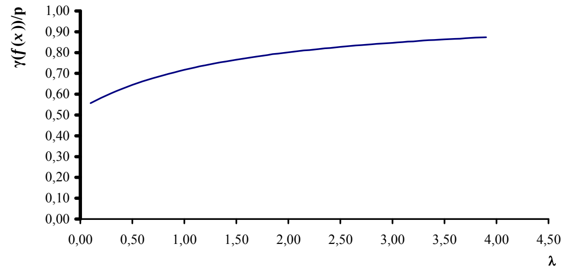


**Fig 3:** The plot of the measure $\gamma(f(x))$ for exponentials $f(x) = \Theta(e^{\lambda x})$.

Let us also put forward several properties of the introduced measure of functions' asymptotic growth — $\gamma(f(x))$:

- — The measure $\gamma(x^k)$ takes the value of $\pi/4$ when $k = \sqrt{2}/2$;
- — The measure $\gamma(e^{\lambda x})$ takes the value of $3\pi/4$ when the exponent $\lambda = \sqrt{2}$;
- — The measure $\gamma(x^k)$ possesses the following notable property: $\gamma(x^{1/\lambda}) + \gamma(e^{\lambda x}) = \pi,$ in particular, $\gamma(x) + \gamma(e^x) = \pi$.

## 9. CLASSIFICATION OF THE ALGORITHMS ON COMPUTATIONAL COMPLEXITY

The usage of the angular measure of functions' asymptotic growth, $\gamma(f(x))$, allows an algorithm classification to be introduced, which differentiates algorithms by the asymptotic growth of their computational complexity (we assume the worst-case complexity). Preserving a common notation $n$ for the algorithm $A$'s input dimension, denoting by $f(n)$ the function of $A$'s computational complexity and assuming a formal transition from $n$ to the real-valued argument $x$ when computing $\gamma(f(x))$ at $x = n$, let us introduce the following set-theoretic class definitions:

1. Class $\pi 0$ (pi zero) — the «fast algorithm» class — contains algorithms whose computational complexity functions belong to the $FZ$ set and have zero measure:
$$\pi 0 = \{ A \mid \gamma(f(n)) = 0 \Leftrightarrow f(n) \in FZ \}.$$

Algorithms of this class are substantially fast with respect to the input length; these are primarily algorithms of poly-logarithmic or logarithmic complexity. For instance, the class contains the algorithm of binary search in an array of sorted keys: its asymptotic complexity estimate is $O(\ln(n))$ [2], and measure $\gamma(\ln(n)) = 0$.

2. Class $\pi P$ — the class of «rational (truly polynomial) algorithms» — contains algorithms, whose complexity functions belong to the $FP$ set:

$$\pi P = \{ A \mid 0 < \gamma(f(n)) < \pi/2 = 0 \Leftrightarrow f(n) \in FP \}.$$

The majority of algorithms that are used in practice and allow one to solve computational problems in a reasonable time, pertain to this class. Note that the class has the property of natural closure. $\pi P$ is a subclass of algorithms defining class $P$ in the computational complexity theory.

3. Класс $\pi L$ — the class of «sub-exponential algorithms» — contains algorithms, whose complexity functions belong to the set $FL$ :

$$\pi L = \{ A \mid \gamma(f(n)) = \pi/2 \Leftrightarrow f(n) \in FL \}.$$

This class is formed by algorithms with super-polynomial yet sub-exponential complexity. Such algorithms are fairly time-consuming, with the corresponding problems primarily belonging to the complexity class $NP$. Nevertheless, for particular tasks such algorithms are put into practice. One of the examples could be the General number sieve method, which is designed for factorization of large composite numbers and is used for direct attacks on the RSA cryptosystem. If $n$ is the number of bits in the number presented for the factorization, then a heuristic estimate of the algorithm's complexity, as given in [13], takes the following form:

$$f(n) = \mathrm{O}\left( e^{O\left( n^{\frac{1}{3}} \cdot (\ln(n))^{\frac{2}{3}} \right)} \right), \text{ and } \gamma(f(n)) = \pi/2 .$$

The symbol $L$ in the name of the class reflects the fact, that the function of exponential logarithm $g(x) = (\ln x)^{\ln x}$ is one of the functions that separate polynomials and exponentials by virtue of Theorem 1.

4. Class $\pi E$ — the class of «truly exponential algorithms» — contains algorithms, whose complexity functions belong to the $FE$ set:

$$\pi E = \{ A \mid \pi/2 < \gamma(f(n)) < \pi \Leftrightarrow f(n) \in FE \}.$$

These are algorithms with exponential complexity, which are currently applicable only for low input dimensions. The potential of real application of such algorithms is tied with the practical implementation of quantum computing. Among the examples of algorithms of the given form are brute-force methods for solving $NP$-complete problems, including SAT, the subset sum problem, the clique problem and others [2]. The problems mentioned have asymptotic complexity estimates of the form $O(2^n), O(n \cdot 2^n), O(n^2 \cdot 2^n)$.

5. Class $\pi F$ — the class of «super-exponential algorithms» — contains algorithms, whose complexity functions belong to the $FF$ set:

$$\pi E = \{ A \mid \gamma(f(n)) = \pi \Leftrightarrow f(n) \in FF \}.$$

This is a class of practically inapplicable algorithms that have a super-exponential complexity, possibly of the factorial form or of the form $\Omega(x^x)$. A brute-force algorithm for the travelling salesman problem has the asymptotic estimate $\Omega(n!)$, and, since $n! = \Gamma(n+1)$, where $\Gamma(x)$ is Euler's gamma function, $\ln(\Gamma(x+1)) \approx x \cdot \ln x - x$, implies $n! \approx e^{n \cdot (\ln n - 1)}$ [14], and since the measure $\gamma\left(e^{x(\ln x - 1)}\right) = \pi$, this algorithms belongs to the class $\pi F$. The class also contains the algorithm for enumerating all spanning trees of a complete $n$-vertices graph, with the asymptotic complexity estimate of $\Omega\left(n^{n-2}\right)$ [2]. The symbol $F$ in the class name reflects the fact that algorithms with factorial complexity belong to the class.

## 10. CONCLUSION

In this study, for the purposes of theoretic analysis and research of computer algorithms:

— an angular measure was introduced for functions' asymptotic growth;
— analysis was performed upon the properties of the measure;
— on the basis of the introduced measure and functional sets, a computational complexity classification of computer algorithms was suggested.

The results obtained may be used in theoretical analysis of computer algorithms' resource efficiency, as well as in practice, for comparative analysis and justification of the selection of rational algorithms.

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