ISSN 0005-1179, Automation and Remote Control, 2006, Vol. 67, No. 12, pp. 1878–1892. © Pleiades Publishing, Inc., 2006.
Original Russian Text © A.A. Tremba, 2006, published in Avtomatika i Telemekhanika, 2006, No. 12, pp. 21–35.

DETERMINATE SYSTEMS

Robust D-decomposition under l_p-bounded Parametric Uncertainties[1]

A. A. Tremba

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Received January 19, 2006

Abstract—Consideration was given to stability of an affine family of uncertain polynomials defined by two real or one complex parameter, the rest of the parameters characterizing indeterminacy. On the plane of family parameters, a domain was established where the uncertain polynomials are stable. The method of robust D-decomposition was used. For the cases where the uncertain parameters are real and bounded in the Euclidean norm or are complex and bounded in the l_p norm, expressions for the boundary of these domains were obtained.

PACS numbers: 02.30.Yy, 02.40.-k, 02.40.Hw

DOI: 10.1134/S0005117906120022

1. INTRODUCTION

Stability of the linear stationary system coincides with stability of its characteristic polynomial. If all parameters of a system and, consequently, the coefficients of the characteristic polynomial are known precisely, then it is possible to judge about stability by performing a single calculation of the roots and check of their location. Verification of stability at modification of the system parameters is the next step in the stability studies. There are two distinct approaches.

The first approach lies in determining a set of points in the space of parameters under which the system is stable. This problem was completely solved by Yu.I. Neimark for the one-dimensional and two-dimensional cases [1, 2]. Moreover, his method of D-decomposition describes decomposition of the entire parameter space into domains with the same number of stable roots of the characteristic polynomial.

The second approach is concerned with the study of system robustness. It checks whether the system will retain stability if its parameters are unknown and belong to a certain set, but are constant. These parameters are often referred to as "uncertainties." The Kharitonov theorem with its counterparts concerning the robust stability of the interval polynomials and the Tsypkin–Polyak locus used under the interval and ellipsoidal constraints on the polynomial coefficients [3–5] are the best-known results of this kind.

In their attempt to merge both approaches. B.T. Polyak and N.P. Petrov in 1991 formulated and for some special cases solved the problem of robust D-decomposition [6] which is as follows: among all parameters at first two real or one complex parameter are specified, the rest of them being uncertain, but norm-bounded. It is required to construct on the plane of the specified parameters a set whose polynomial is stable under any admissible values of the uncertain parameters. We note that this decomposition of the parameters into two classes is characteristic of design of the robust controllers of a given structure.

[1] This work was supported in part by the Russian Foundation for Basic Research, project nos. 05-01-00114, 05-08-01177, and the Program for Basic Research of the Presidium of the Russian Academy of Sciences, project no. 19.

The present paper generalizes the method of robust D-decomposition to the affine polynomial family. The uncertain parameters are real and bounded in the Euclidean norm or complex and bounded in the l_p-norm. The problem of robust D-decomposition is formulated and solved in Section 2 for two specified real parameters and in Section 3 for one complex parameter. Examples of solutions are given in Section 4. In summary the conclusions are set forth, and an approach to more complicated problems is presented.

2. ROBUST D-DECOMPOSITION FOR TWO REAL PARAMETERS

2.1. Parametric Polynomial Family

We recall the principles of D-decomposition [1, 2]. Let a polynomial of degree n be linearly dependent on two real parameters τ and ν making up the family

$$\widetilde{G}(s,\tau,\nu) \doteq \tau\widetilde{P}(s) + \nu\widetilde{Q}(s) + \widetilde{R}(s), \tag{1}$$

where $\widetilde{P}(s)$, $\widetilde{Q}(s)$, and $\widetilde{R}(s)$ are fixed polynomials.

Polynomial (1) is referred to as $\mathbb{D}$-stable if all its n roots are stable, that is, lie within the stability domain $\mathbb{D}^{stab} \subset \mathbb{C}$. The complement to the closure $\mathbb{D}^{stab}$ will be denoted by $\mathbb{D}^{unst}$.

The open left half-plane $\{s : \operatorname{Re} s < 0\}$ and the exterior of the unit circle $\{s : |s| > 1\}$ are the stability domains, respectively, for the continuous-time and discrete-time systems [5]. We may simply distinguish stability in the continuous and discrete sense.

Definition 1. For the polynomial $\widetilde{G}$ like (1), by the D-decomposition in the parameters (τ,ν) for the stability domain $\mathbb{D}^{stab}$ is meant the separation of the parameter space (τ,ν) into the sets $D(k)$ $(0 \leqslant k \leqslant n)$ defined by the number of stable roots:

$$D(k) = \left\{(\tau,\nu) : \widetilde{G}(s,\tau,\nu) \text{ has } k \text{ roots in } \mathbb{D}^{stab} \text{ and } n-k \text{ roots in } \mathbb{D}^{unst}\right\}.$$

We consider below stability in the continuous sense, provided that it is not stipulated otherwise. The discrete case leads to the same results, some distinctions being mentioned in Remark 3 at the end of this subsection.

Let us see what happens to the polynomial roots if the parameters (τ,ν) go from the domain $D(k)$ to the domain $D(i)$ $(k \neq i)$. If at that the degree of the polynomial $\widetilde{G}(s,\tau,\nu)$ remains invariable, then its roots depend continuously on the parameters and the transition from $D(k)$ to $D(i)$ is accompanied by the transition of several roots through the imaginary axis. By definition, those parameters (τ,ν) for which the degree $\widetilde{G}$ becomes smaller than n do not belong to any domain of D-decomposition.

Therefore, at least one of the following two conditions is satisfied for each point (τ,ν) on the boundary $\Gamma = \bigcup_{k=0\ldots n} \partial D(k)$ between the domains of D-decomposition:

(1) The degree $\widetilde{G}$ drops

$$\tau\widetilde{p}_n + \nu\widetilde{q}_n + \widetilde{r}_n = 0, \tag{2}$$

where $\widetilde{p}_n$, $\widetilde{q}_n$, $\widetilde{r}_n$ are the coefficients of the polynomials $\widetilde{P}$, $\widetilde{Q}$, and $\widetilde{R}$ for s^n; some of them being, possibly, zero.

(2) $\widetilde{G}$ has a purely imaginary root:

$$\widetilde{G}(s,\tau,\nu) = 0,\ \operatorname{Re} s = 0. \tag{3}$$

The last equation is called the basic equation of D-decomposition or simply the basic equation. It demonstrates the *principle of elimination of zero*: only those parameters for which polynomial (1) does not vanish on the imaginary axis belong to the domains $D(k)$.

The D-decomposition can be realized by specifying all those parameters (τ, ν) for which the root of the polynomial $\widetilde{G}(s, \tau, \nu)$ lies on the imaginary axis. The necessary and sufficient conditions for membership of the point (τ, ν) to the boundary of D-decomposition are given by Eqs. (2) and (3) [2].

It is only natural to take the real variable ω and select all points of the imaginary axis by means of the substitution $s = j\omega$ ($s = e^{j\omega}, \omega \in [0, 2\pi)$ for the discrete case). Then, the basic equation goes over to

$$G(\omega, \tau, \nu) \doteq \tau P(\omega) + \nu Q(\omega) + R(\omega) = 0, \quad \omega \in (-\infty, +\infty), \tag{4}$$

where $P(\omega) \doteq \widetilde{P}(j\omega)$, $Q(\omega) \doteq \widetilde{Q}(j\omega)$, $R(\omega) \doteq \widetilde{R}(j\omega)$.

This complex equation corresponds to a system of two equations (for the imaginary and real parts) that can be conveniently set down in the vector form as

$$T(\omega)x = -r(\omega), \tag{5}$$

where

$$T(\omega) \doteq \begin{bmatrix} \operatorname{Re} P(\omega) & \operatorname{Re} Q(\omega) \\ \operatorname{Im} P(\omega) & \operatorname{Im} Q(\omega) \end{bmatrix}, \quad x \doteq \begin{bmatrix} \tau \\ \nu \end{bmatrix}, \quad r(\omega) \doteq \begin{bmatrix} \operatorname{Re} R(\omega) \\ \operatorname{Im} R(\omega) \end{bmatrix}. \tag{6}$$

For a fixed ω, the set of solutions of system (5) can be

(1) a point if $\det T(\omega) \neq 0$;

(2) an empty set if $\operatorname{rank} T(\omega) < \operatorname{rank} [T(\omega)\ r(\omega)]$;

(3) a straight line if $\operatorname{rank} T(\omega) = \operatorname{rank} [T(\omega)\ r(\omega)] = 1$;

(4) the entire plane if $\operatorname{rank} [T(\omega)\ r(\omega)] = 0$.

The union of these sets in all $\omega \in (-\infty, +\infty)$ and also solution of Eq. (2) provide the set Γ performing D-decomposition. As a rule, Γ consists of one curve and more than one straight line corresponding to Case 3. If the fourth case is realized at least for one ω, then all sets $D(k)$ are empty (the polynomials $P(\omega), Q(\omega), R(\omega)$ have a general real root; in distinction to the robust variant, this case can be readily identified).

Remark 1. If the polynomials $\widetilde{P}(s), \widetilde{Q}(s), \widetilde{R}(s)$ have real coefficients, then it is sufficient that the variable ω in the basic Eq. (4) belongs to the interval $[0, \infty)$, $\omega = 0$ defining a special line. Without loss of generality we consider namely such polynomials.

2.2. Family of the Uncertain Polynomials

We introduce an additional vector parameter $a \in \mathbb{R}^\ell$ (or $a \in \mathbb{C}^\ell$) in the polynomial $\widetilde{G}$ of the form (1) and consider the set of polynomials $\{\widetilde{G}(s, \tau, \nu, a) : a \in \mathcal{A}\}$, where $\mathcal{A}$ is some subset of the parameter space.

In the main case, we assume that a is involved only in one polynomial, namely, in $\widetilde{R}$:

$$\widetilde{G}(s, \tau, \nu, a) = \tau \widetilde{P}(s) + \nu \widetilde{Q}(s) + \widetilde{R}(s, a). \tag{7}$$

We note nonequivalence of the parameters τ, ν, and a. The two first scalars (or one complex parameter, see Section 3) are regarded as the controlled parameters. The fact that there are only two parameters allows one to make efficient use of the graphic methods; however, for the same reason the number of the controlled parameters cannot exceed two. The parameters a_i and the set $\mathcal{A}$ characterize the error description, time-constant perturbation, or uncertainty of the nominal system. In what follows, the term "uncertainty" will be applied to a proper.

We make an explanation about the ambiguity of notation: on the one hand, $\widetilde{G}(s, \tau, \nu, a)$ may be regarded as the set $\{\widetilde{G}_{\tau,\nu}(s, a) : a \in \mathcal{A}\}$ of polynomials that are parametric in (τ, ν); on the other hand, it is possible to consider the parametric family (in τ, ν) of uncertain polynomials $\widetilde{G}_{\mathcal{A}}(s, \tau, \nu)$ in the context of the problem of robust design and boundedness of a. We recall that by the uncertain polynomial is meant a set of polynomials, in the case under consideration, $\widetilde{G}_{\mathcal{A}}(s) = \{\widetilde{G}(s, a) : a \in \mathcal{A}\}$. In the context of applying the robust D-decomposition to the controller design, we make use of the last variant.

Definition 2. For the polynomial family $\widetilde{G}(s, \tau, \nu, a)$, by the robust D-decomposition meant are the sets $D(k)(0 \leqslant k \leqslant n)$ in each of which the numbers of stable and unstable roots are the same for all admissible uncertainties:

$$D(k) = \left\{ (\tau, \nu) : \widetilde{G}(s, \tau, \nu, a) \quad \begin{matrix} \text{has } k \text{ roots in the left half-plane} \\ \text{and } n-k \text{ roots in the right half-plane } \forall a \in \mathcal{A} \end{matrix} \right\},$$

that is, in each domain $D(k)$ the uncertain polynomial $\widetilde{G}_{\mathcal{A}}(s, \tau, \nu)$ has the degree n and precisely k stable roots.

By analogy with the conventional D-decomposition, we consider the set Υ_s corresponding to reduction of the polynomial degree (compare with Eq. (2)):

$$\Upsilon_s = \{(\tau, \nu) : \tau \widetilde{p}_n + \nu \widetilde{q}_n + \widetilde{r}_n(a) = 0,\ a \in \mathcal{A}\}.$$

Stated differently, Υ_s is the solution of the inclusion

$$\tau \widetilde{p}_n + \nu \widetilde{q}_n \in -\widetilde{r}_{\mathcal{A}n}, \tag{8}$$

where $\widetilde{r}_{\mathcal{A}n} \doteq \{\widetilde{r}_n(a) : a \in \mathcal{A}\}$ is the set of values of the coefficient at s^n of the uncertain polynomial $\widetilde{R}_{\mathcal{A}}$.

By means of the principle of elimination of zero we get the basic equation of robust D-decomposition (counterpart of (3), the substitution $s = j\omega$ in the continuous case)

$$\tau P(\omega) + \nu Q(\omega) + R(\omega, a) = 0, \quad a \in \mathcal{A}, \tag{9}$$

which is a family of equations in the parameter ω. We put it down in terms of sets:

$$\begin{gathered} \tau P(\omega) + \nu Q(\omega) \in -\mathcal{R}_{\mathcal{A}}(\omega), \\ \mathcal{R}_{\mathcal{A}}(\omega) \doteq \{R(\omega, a) : a \in \mathcal{A}\}, \quad \omega \in [0, \infty), \end{gathered} \tag{10}$$

and denote by $\Upsilon(\omega)$ its solution for (τ, ν). The totality of these solutions in all ω makes up the set $\Upsilon_\Omega \doteq \bigcup_{\omega \in [0,\infty)} \Upsilon(\omega)$. Geometrically speaking, $\Upsilon(\omega)$ is a planar figure, and Υ_Ω is the area swept by it.

Lemma 1. *The sets $D(k)$ and $\Upsilon \doteq \Upsilon_\Omega \bigcup \Upsilon_s$ exhaust the space of parameters (τ, ν):*

$$\begin{gathered} \Upsilon \bigcap D(k) = \emptyset, \quad 0 \leqslant k \leqslant n, \\ D(i) \bigcap D(k) = \emptyset, \quad i \neq k, \\ \Upsilon \bigcup_{0 \leqslant k \leqslant n} D(k) = \mathbb{R}^2. \end{gathered} \tag{11}$$

The proof follows directly from Definition 2 and Eqs. (8) and (9).

Therefore, the set Υ which divides the domains of the robust D-decomposition $D(k)$ is made up in part by the set Υ_s and the area Υ_Ω swept by the geometrical figure $\Upsilon(\omega)$. Speaking nonrigorously, the set Υ is obtained by "smearing" the curve Γ of the ordinary D-decomposition into a band (see also the examples in Section 4).

In contrast to the case of uncertainty-free D-decomposition, it is not the set Υ itself but its boundary $\partial\Upsilon$ that bounds the domains of robust D-decomposition. Therefore, the problem is solved by constructing this boundary and specifying among the domains that of stability.

Examples where $\mathcal{R}(\omega)$ is a polygon and circle corresponding to the interval norm for the real and complex uncertainties can be found in [6]. In the next section we consider a more general case where $\mathcal{R}(\omega)$ is an ellipse.

Besides the family of the form (7), it is possible to consider the case where the uncertain polynomial is with one of the parameters used to construct the D-decomposition. We will prove that this case comes to that considered above. Without loss of generality we consider the parameter τ and, correspondingly, the uncertainty of the form $P(\omega, a), a \in \mathcal{A}$. The points $\nu = -\dfrac{R(\omega)}{Q(\omega)} \in \mathbb{R}$ represent the solution of the basic equation

$$\tau P(\omega, a) + \nu Q(\omega) + R(\omega) = 0, \quad a \in \mathcal{A}, \tag{12}$$

on the line $\tau = 0$ (we recall that $R(\omega)$ and $Q(\omega)$ are the complex-valued polynomials). For $\tau \neq 0$, we change the variables $\widehat{\tau} \doteq \dfrac{1}{\tau}$, $\widehat{\nu} \doteq \dfrac{\nu}{\tau}$, and after renotation Eq. (12) is rearranged in the aforementioned form (9):

$$\widehat{\tau} R(\omega) + \widehat{\nu} Q(\omega) + P(\omega, a) = 0, \quad a \in \mathcal{A}.$$

We note that the transformation $(\tau, \nu) \leftrightarrow (\widehat{\tau}, \widehat{\nu})$ drives the straight (in particular, special) lines to straight lines.

The cases where there is uncertainty simultaneously in two or more polynomials $P(\omega), Q(\omega), R(\omega)$ technically are much more difficult, but similar in the procedural terms. A method of determination of the sufficient results is mentioned in conclusion.

2.3. l_p-bounded Uncertainties

Let the uncertain polynomial $\widetilde{G}$ by defined by the affine family

$$\widetilde{G}(s, \tau, \nu, a) = \tau \widetilde{P}(s) + \nu \widetilde{Q}(s) + \widetilde{R}_0(s) + \sum_{i=1}^{\ell} a_i \widetilde{R}_i(s), \quad s = j\omega, \tag{13}$$

where $a = (a_i)$, $i = 1, \ldots, \ell$ is the vector of real or complex parameters bounded in the l_p-norm by a positive number γ: $\|a\|_p < \gamma$. Here and below, the symbol of norm without a subscript denotes the Euclidean norm, and with it, the l_p-norm.

Let us consider the set

$$\mathcal{R}_{\mathcal{A}}(\omega) = \left\{ R_0(\omega) + \sum_{i=1}^{\ell} a_i R_i(\omega) : \|a\|_p \leqslant \gamma,\ a_i \in \mathbb{F} \right\}, \tag{14}$$

where the parameter field $\mathbb{F}$ is either $\mathbb{R}$ (the real case) or $\mathbb{C}$ (the complex case); the norm index p belongs to the interval $[1, \infty)$. We fix ω. Then, $\mathcal{R}_{\mathcal{A}}(\omega)$ is the set of values of the linear function in the parameters $a \in \mathcal{A}$; here and below, the dependence on ω is omitted if ω is fixed and does not affect the results.

Lemma 2. *The set $\mathcal{R}_\mathcal{A}$ of the form* (14) *is*

(1) *a circle if $a_i \in \mathbb{C}$, $p \in [1, \infty)$,*

$$\mathcal{R}_\mathcal{A} = \left\{ z : \|z - r_0\| \leqslant \gamma \|R\|_q, \ z \in \mathbb{R}^2 \right\}, \tag{15}$$

where $q = p/(p-1)$, the vector $R = (R_i)$, $r_0 \doteq -[\operatorname{Re} R_0, \operatorname{Im} R_0]^{\mathrm{T}}$;

(2) *an ellipse if $a_i \in \mathbb{R}$, $p = 2$,*

$$\mathcal{R}_\mathcal{A} = \begin{cases} \left\{ r : (r - r_0)^{\mathrm{T}} M^{-1} (r - r_0) \leqslant \gamma, \ r \in \mathbb{R}^2 \right\}, \ \det(M) \neq 0 \\ \left\{ r_0 + \chi w : |\chi| \leqslant \gamma\sqrt{\lambda}, \chi \in \mathbb{R} \right\}, \ \det(M) = 0, \end{cases} \tag{16}$$

where used are the matrix

$$M = \begin{bmatrix} \sum\limits_{i=1}^{\ell} (\operatorname{Re} R_i)^2 & \sum\limits_{i=1}^{\ell} \operatorname{Re} R_i \operatorname{Im} R_i \\ \sum\limits_{i=1}^{\ell} \operatorname{Re} R_i \operatorname{Im} R_i & \sum\limits_{i=1}^{\ell} (\operatorname{Im} R_i)^2 \end{bmatrix}, \tag{17}$$

its normalized eigenvector w corresponding to the maximal eigenvalue λ, and $r_0 \doteq -[\operatorname{Re} R_0, \operatorname{Im} R_0]^{\mathrm{T}}$.

Here and below, the complex plane is identified with $\mathbb{R}^2$. The lemma is proved in the Appendix.

In the general case, the ellipse is defined by a nonnegative definite matrix M, the center r_0, and the number ρ:

$$\mathcal{R}_\mathcal{A}(\omega) = \left\{ r : (r - r_0(\omega))^{\mathrm{T}} M(\omega)^{-1} (r - r_0(\omega)) \leqslant \rho(\omega)^2 \right\}.$$

With regard for the notation (6), the solution of the basic equation of the robust D-decomposition (10) is as follows:

$$\Upsilon(\omega) = \left\{ x : (T(\omega)x - r_0(\omega))^{\mathrm{T}} M(\omega)^{-1} (T(\omega)x - r_0(\omega)) \leqslant \rho(\omega)^2 \right\}. \tag{18}$$

If the matrix $T(\omega)$ is nondegenerate, the set of $\Upsilon(\omega)$ satisfying (10) is an ellipse as well; otherwise, $\Upsilon(\omega)$ is empty or is a band. These cases will be discussed separately.

An expression for the boundaries Υ_Ω, which also are the boundaries of the domains of D-decomposition $D(k)$, can be readily established using the following data of the differential geometry [7]. The points of the envelope of the parametric family of curves defined by the equation $F(x, y, k) = 0$ (x, y are the coordinates and k is the parameter), must satisfy the equation system

$$\begin{cases} F(x, y, k) = 0 \\ \dfrac{\partial F(x, y, k)}{\partial k} = 0. \end{cases} \tag{19}$$

Let us formulate a theorem about construction of the boundaries of the robust D-decomposition for the real parameters.

Theorem 1. *If the matrices $T(\omega)$ and $M(\omega)$ are nondegenerate, then the points of the boundary of the robust D-decomposition for the polynomial $\widetilde{G}(s, \tau, \nu, a)$ of the form* (13) *are the solutions of the system*

$$\begin{cases} (Tx - r_0)^{\mathrm{T}} \widehat{M} (Tx - r_0) = \rho^2 \det(\widehat{M}) \\ x^{\mathrm{T}} (T^{\mathrm{T}} \widehat{M} T)'_\omega x - 2(r_0^{\mathrm{T}} \widehat{M} T)'_\omega x + (r_0^{\mathrm{T}} \widehat{M} r_0)'_\omega = \left(\rho^2 \det(\widehat{M}) \right)'_\omega, \end{cases} \tag{20}$$

for the vector $x \doteq (\tau, \nu)^{\mathrm{T}}$, $\omega \in [0, \infty)$.

The notation of (6) *is used for the matrix* T *and the vector* r_0*, the matrices* $\widehat{M}$ *and* M *are related by*

$$M^{-1} \doteq \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{12} & m_{11} \end{bmatrix} \doteq \frac{1}{\det(M)} \widehat{M} = \frac{1}{\det\left(\widehat{M}\right)} \widehat{M}.$$

By Lemma 2, M *is either the identity matrix or has the form* (17)*, and* ρ *is equal to* $\|R\|_q$ *or* 1 *depending on the uncertainty* a*.*

The proof follows immediately from the substitution of the expression for the ellipse family (18) in the envelope Eqs. (19).

In the nondegenerate case, the established system of two equations (each of which has at most the order of two) of two variables has at most four solutions providing the envelopes of the ellipse family (18) which are part of the boundary of Υ_Ω. The remaining part of the boundary of Υ_Ω is defined by the same ω for which one or two matrices $T(\omega)$ and $M(\omega)$ degenerate or one of the derivatives does not exist in (20).

Let us consider in more detail the special cases of degeneration of the matrices T and M.

(1) $T(\omega) = 0$, $\det M(\omega) = 0$. If there is a number χ such that $r_0 + \chi w = 0$, $|\chi| \leqslant \rho\sqrt{\lambda}$, where w is a nonzero normalized eigenvector of the matrix M corresponding to its maximum eigenvalue λ, then $\Upsilon(\omega) = \mathbb{R}^2$ and $\partial\Upsilon = \{\emptyset\}$. If there exists such a number at least for one $\omega \in [0, \infty)$, then all the domains $D(k)$ will be empty as in the case of D-decomposition for the nonperturbed polynomials. If there is no such χ, then $\Upsilon(\omega) = \{\emptyset\}$.

(2) $T(\omega) = 0$, $\det M(\omega) \neq 0$. If $r_0^{\mathrm{T}} M^{-1} r_0 > \rho^2$, then it follows from (18) that $\Upsilon(\omega) = \{\emptyset\}$, and $r_0(\omega)^{\mathrm{T}} M(\omega)^{-1} r_0(\omega) \leqslant \rho(\omega)^2$ implies that $\Upsilon(\omega) = \mathbb{R}^2$. As in the previous case, at least one ω for which $\Upsilon(\omega) = \mathbb{R}^2$ suffices for emptiness of all $D(k)$.

(3) rank $T(\omega) = 1$, $\det M(\omega) \neq 0$. In this case, Eq. (18) describes the band

$$\Upsilon(\omega) = \{x : x = \mu T f + \eta v,\ \mu \in [\mu_1, \mu_2],\ \eta \in (-\infty, +\infty)\},$$

where f and v are the singular vectors of the matrix $T(\omega)$ corresponding, respectively, to the nonzero and zero singular values (they make up the basis) and μ_1 and μ_2 are the real solutions of the equation

$$\left(f^{\mathrm{T}} T^{\mathrm{T}} M^{-1} T f\right) \mu^2 - 2\left(r_0^{\mathrm{T}} M^{-1} T f\right) \mu + r_0^{\mathrm{T}} M^{-1} r_0 - \rho^2 = 0.$$

If there are no real roots, then the set $\Upsilon(\omega)$ is empty.

(4) $\det T(\omega) \neq 0$, $\det M(\omega) = 0$. The ellipse $\mathcal{R}_{\mathcal{A}}$ degenerates into the segment (16), the same happens also to $\Upsilon(\omega)$:

$$\Upsilon(\omega) = \left\{T^{-1}(r_0 + \chi w) : |\chi| \leqslant \rho\sqrt{\lambda}\right\},$$

where w is the normalized eigenvector of the matrix M corresponding to its maximum eigenvalue (possibly, zero, which comprises the case of $M = 0$).

(5) rank $T(\omega) = 1$, $\det M(\omega) = 0$. Let f and v be the singular vectors of the matrix $T(\omega)$ corresponding, respectively, to the nonzero and zero eigenvalues, the first vector being normalized so that the norm of the vector $u \doteq Tf$ is equal to 1; w is the normalized eigenvector of the matrix $M(\omega)$ corresponding to its maximum eigenvalue λ. Three variants are possible here: empty set, straight line, and band; $\|$ and $\nparallel$ denoting here collinearity and noncollinearity of the vectors:

—$u \parallel w \parallel r_0$ (band):

$$\Upsilon(\omega) = \{\mu u + \eta v : \mu \in [\mu_1, \mu_2],\ \eta \in (-\infty, +\infty)\},$$

where $\mu_{1,2} = (r_0, u) \pm \rho\sqrt{\lambda}$ and the subscripts $1, 2$ concern the vector components;

—$u \parallel w, r_0 \nparallel u$ (empty set), $\Upsilon(\omega) = \{\emptyset\}$;

—$u \nparallel w$, $(u_1 r_{02} - u_2 r_{01}) > \rho\sqrt{\lambda}(u_1 w_2 - u_2 w_1)$ (empty set), $\Upsilon(\omega) = \{\emptyset\}$;

—$u \nparallel w$, $(u_1 r_{02} - u_2 r_{01}) \leqslant \rho\sqrt{\lambda}(u_1 w_2 - u_2 w_1)$ (straight line):

$$\Upsilon(\omega) = \left\{\mu u + \eta v : \mu = \frac{w_1 r_{02} - w_2 r_{01}}{u_1 w_2 - u_2 w_1},\ \eta \in (-\infty, +\infty)\right\}.$$

Finally, if for some ω the derivative of the second equation of system (20) is not defined (for example, $\rho(\omega)$ may be nonsmooth), then the right and left limits in ω are used to construct the envelope. An arc of the ellipse (18) corresponding to this value of ω can be added to the common boundary $\partial\Upsilon_\Omega$.

Remark 2. If the parameter vector a is bounded in the weighted norm $\|(a_i)\|^\alpha \doteq \|(a_i/\alpha_i)\|$, $\alpha_i > 0$, then replacement of the polynomials $R_i(\omega)$ by $\widehat{R}_i(\omega) \doteq \alpha_i R_i(\omega)$ reduces this case to the previously considered one. Obviously,

$$\left\{\sum_{i=1}^{\ell} a_i R_i(\omega) : \|a\|^\alpha \leqslant \gamma\right\} = \left\{\sum_{i=1}^{\ell} a_i \widehat{R}_i(\omega) : \|a\| \leqslant \gamma\right\}.$$

Remark 3. In the case of discrete stability, (i) the substitution $s = e^{j\omega}$ is used and (ii) variation of the degree of the polynomial does not lead to instability. Therefore, the condition (8) for reduction of the polynomial degree may be discarded, and its corresponding set Υ_s may be regarded as empty.

3. ROBUST D-DECOMPOSITION FOR THE COMPLEX PARAMETER

The case of robust D-decomposition for one complex parameter π

$$\widetilde{G}(s, \pi, a) = \pi\widetilde{D}(s) + \widetilde{R}_0(s) + \sum_{i=1}^{\ell} a_i \widetilde{R}_i(s) \tag{21}$$

is reduced to the case (13) of two real parameters by means of the substitution $\tau \doteq \operatorname{Re}\pi$, $\nu \doteq \operatorname{Im}\pi$, $\widetilde{P}(s) \doteq \widetilde{D}(s)$, $\widetilde{Q}(s) \doteq j\widetilde{D}(s)$.

For the complex parameter π, it is only natural to consider only the complex uncertainties like $\|a\|_p \leqslant \gamma$ ($a \in \mathbb{C}^\ell, p \in [1, \infty)$). Then, according to Lemma 2, the set of values of the family $\left\{\sum a_i \widetilde{R}_i(s) : \|a\|_p \leqslant \gamma\right\}$ is the circle (15). The following theorem defines explicitly the boundaries of the domains $D(k)$.

Theorem 2. *The boundaries $\partial\Upsilon$ of the robust D-decomposition for the complex parameter π of the polynomial*

$$\pi\widetilde{D}(s) + \widetilde{R}_0(s) + \sum_{i=1}^{\ell} a_i \widetilde{R}_i(s) \tag{22}$$

with uncertainty $a \in \mathbb{C}^\ell$, $\|a\|_p \leqslant \gamma$ in the continuous sense ($s = j\omega$) are described as follows:

$$\begin{aligned} &\partial\Upsilon = \{\emptyset\}, && \text{if } \exists\, \omega \in [0, \infty) : D(\omega) = 0,\ |R_0(\omega)| \leqslant \rho(\omega), \\ &\partial\Upsilon \subset \partial\Upsilon_\Omega \bigcup \partial\Upsilon_s, && \text{otherwise}, \end{aligned} \tag{23}$$

where

$$\partial\Upsilon_\Omega = \left\{\pi_0(\omega) - \rho(\omega)d\pi_0(\omega)\left[d\rho(\omega) \pm j\sqrt{1-d\rho(\omega)^2}\right] : \begin{array}{c}|\pi_0(\omega)'| \geqslant |\rho(\omega)'| > 0,\\ D(\omega) \neq 0, \omega \in \mathbb{R}\end{array}\right\}$$

$$\bigcup\left\{\pi : |\pi - \pi_0(\omega)| = \rho(\omega) : \pi_0(\omega)' = |\rho(\omega)'| = 0\right\},$$

$$\partial\Upsilon_s = \left\{\pi : \left|\pi\widetilde{d}_n + \widetilde{r}_{0n}\right| = \|\widetilde{r}_n\|_q\right\},$$

where $\pi_0(\omega) \doteq -\dfrac{R_0(\omega)}{D(\omega)}$, $\rho(\omega) \doteq \|R\|_q$, $q = \dfrac{p}{p-1}$, $d\pi_0(\omega) \doteq \dfrac{\pi_0(\omega)'}{|\pi_0(\omega)'|}$, $d\rho(\omega) \doteq \dfrac{\rho(\omega)'}{|\pi_0(\omega)'|}$, $\widetilde{r}_n = (\widetilde{r}_{in})$; $\widetilde{d}_n$, $\widetilde{r}_{0n}$ *and* $\widetilde{r}_{in}$ *are the coefficients at* s^n *of the polynomials* $\widetilde{D}(s)$, $\widetilde{R}_0(s)$, *and* $\widetilde{R}_i(s)$, $D(\omega) \doteq \widetilde{D}(j\omega)$, $R_i(\omega) \doteq \widetilde{R}_i(j\omega)$.

The theorem is proved in the Appendix.

The next case where the uncertainty exists in the polynomial $\widetilde{D}(s)$ is already irreducible to the real case. However, the basic equation

$$\pi\left(D_0(\omega) + \sum_{i=1}^{\ell} a_i D_i(\omega)\right) + R(\omega) = 0, \quad \omega \in [0,\infty), \quad \|a\|_p \leqslant 1, \tag{24}$$

can be solved either by passing to $\widehat{\pi} \doteq 1/\pi$ similar to the case of two real parameters or using the circular arithmetics [8]. We present the second method.

Definition 3. By the circular number $\mathcal{C}(c,\varrho)$ is meant the set of complex numbers

$$\mathcal{C}(c,\varrho) = \{z : |z-c| \leqslant \varrho, z, c \in \mathbb{C}\}.$$

The operations over the circular numbers are understood as the set calculus:

$$b_1\mathcal{C}(c_1,\varrho_1) + b_2\mathcal{C}(c_2,\varrho_2) = \{b_1z_1 + b_2z_2 : z_1 \in \mathcal{C}(c_1,\varrho_1), z_2 \in \mathcal{C}(c_2,\varrho_2)\}, b_1, b_2 \in \mathbb{C},$$

$$\frac{\mathcal{C}(c_1,\varrho_1)}{\mathcal{C}(c_2,\varrho_2)} = \mathcal{C}(c_1,\varrho_1)\mathcal{C}^{-1}(c_2,\varrho_2) = \left\{\frac{z_1}{z_2} : z_1 \in \mathcal{C}(c_1,\varrho_1), z_2 \in \mathcal{C}(c_2,\varrho_2)\right\}, \varrho_2 < |c_2|.$$

The solutions of Eq. (24) are as follows:

$$\Upsilon(\omega) = -R(\omega)\mathcal{C}^{-1}(D_0(\omega), \|D(\omega)\|_q), \quad \omega \in [0,\infty).$$

Inversion of the circular number provides also the circular number $\mathcal{C}^{-1}(c,\varrho) = (|c|^2 - \varrho^2)^{-1}\mathcal{C}(\overline{a},\varrho)$, where the overline denotes complex conjugation. The set Υ_Ω is generated by moving the circle $(|D_0(\omega)|^2 - \|D(\omega)\|_q^2)^{-1}\mathcal{C}(-R_0(\omega)\overline{D}_0(\omega), \|D(\omega)\|_q)$, $\omega \in [0,\infty)$. Theorem 2 can be used to determine its boundary.

4. EXAMPLES

Example 1. Let us consider the use of the robust D-decomposition for design of the PI controller. There exists an uncertain system (based on Example 1 from [9]) with the transfer function $\widetilde{G}$:

$$\begin{aligned}&\widetilde{G}(s,b_1,b_2) = \frac{(s-1)(s-2)}{(s+1)(b_1s^2 + b_2s + 1)},\\ &(b_1-1)^2 + 0.25(b_2-1)^2 \leqslant 0.01; \quad b_1, b_2 \in \mathbb{R},\end{aligned} \tag{25}$$

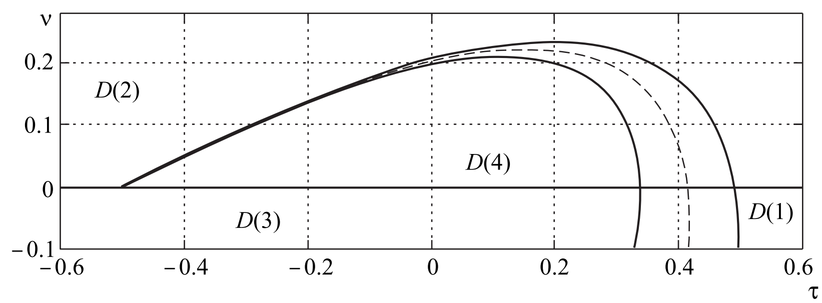

Fig. 1. Robust D-decomposition for system (25).

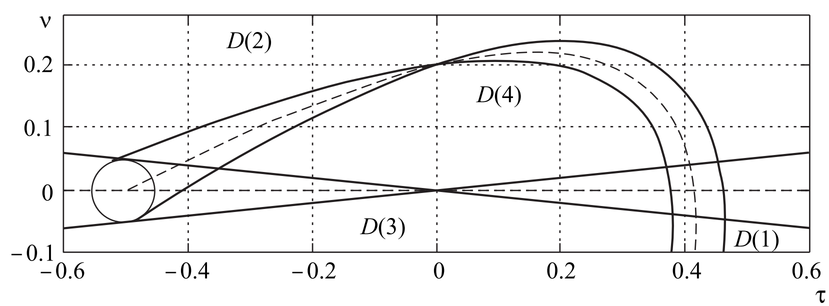

Fig. 2. Robust D-decomposition for system (27).

where b_1, b_2 are uncertain parameters. It is required to determine the set of all PI controllers like $C(s) = \tau + \dfrac{\nu}{s}$ stabilizing the system.

We renotate the uncertain parameters and come to the standard form (13) of the uncertain characteristic polynomial:

$$\tau s(s-1)(s-2) + \nu(s-1)(s-2) + s(s+1)(s^2+s+1) + 0.1a_1s^3(s+1) + 0.2a_2s^2(s+1) = 0, \quad (26)$$
$$\|a\|_2 \leqslant 1,\ a_1 \doteq 10(b_1-1),\ a_2 \doteq 5(b_2-1).$$

For $\omega = 0$, there is a single special line $\tau = 0$. The stability domain $D(4)$ is shown in Fig. 1 where the dashed line represents the curve of D-decomposition of the system without uncertainties ($b_1 = 1, b_2 = 1$).

Example 2. The case where the system is assumed to be defined precisely and the uncertainty exist in the controller parameters (namely, in the proportional and integral units). The transfer function of the closed-loop system is the same as in the example:

$$\widetilde{G}(s) = \frac{(s-1)(s-2)}{(s+1)(s^2+s+1)},$$
$$C(s, b_1, b_2) = \tau\left(1 + b_1 + \frac{b_2}{s}\right) + \frac{\nu}{s}, \quad (27)$$
$$b_1^2 + b_2^2 \leqslant 0.01; \quad b_1, b_2 \in \mathbb{R}.$$

Similar to the previous case, we compile the basic equation of the robust D-decomposition

$$\tau(s + 0.1a_1s + 0.1a_2)(s-1)(s-2) + \nu(s-1)(s-2) + s(s+1)(s^2+s+1) = 0,\ s = j\omega, \quad (28)$$
$$\|a\|_2 \leqslant 1,\ a_1 \doteq 10b_1,\ a_2 \doteq 10b_2.$$

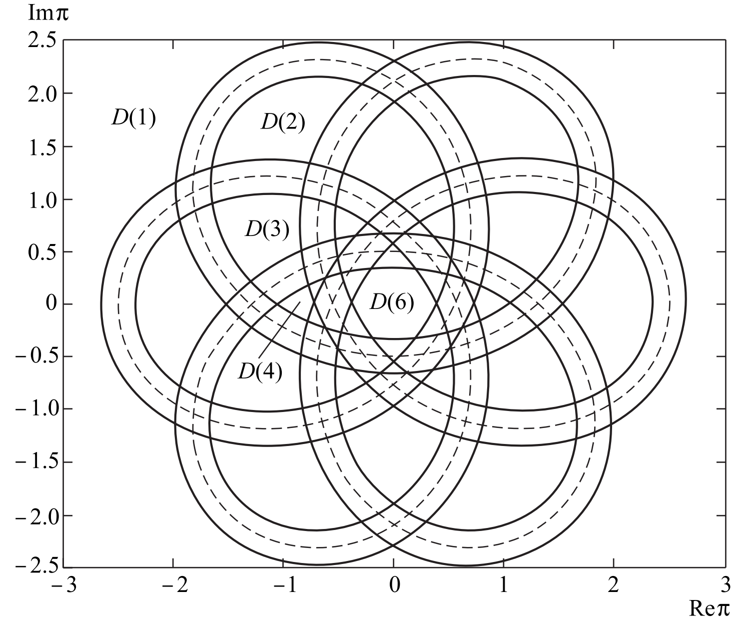

Fig. 3. Robust D-decomposition for the complex parameter.

As can be seen, it has the type of (12). By means of the transformations $\widehat{\tau} \doteq 1/\tau$, $\widehat{\nu} \doteq \nu/\tau$ (see the end of Section 2.2), the robust D-decomposition was constructed (Fig. 2, the stability domain $D(4)$). The sector $|\tau| < 10|\nu|$ corresponds to the special band $|\widehat{\nu}| \leqslant 0.1$ in the auxiliary coordinates $(\widehat{\tau}, \widehat{\nu})$ for $\omega = 0$.

Example 3. Let us consider the robust D-decomposition of the discrete polynomial (see Example 2 from [10]):

$$\begin{gathered} z^6(1 + 0.1a_1) + \pi z^5 + 1.5 + 0.1a_2 = 0; \quad z = e^{j\omega}, \\ |a_i| \leqslant 1, \quad a_i \in \mathbb{C}, \quad i = 1, 2. \end{gathered} \tag{29}$$

One can see from the D-decomposition in Fig. 3 that, in distinction to the original polynomial $a_1 = 0$, $a_2 = 0$, the presence of uncertainty does not help in selecting the parameter π so as to make exactly one root of polynomial (29) to lie within the unit circle (the D-decomposition without uncertainties is denoted by the dashed line). The special lines and bands lack; in other respects, the construction resembles the continuous case: the boundaries "follow" those of the nominal D-decomposition of the domain $D(k)$ (in particular, the stability domain $D(6)$) decreased as compared with the D-decomposition without uncertainties.

5. CONCLUSIONS

The paper described the robust D-decomposition in two real parameters for the affine family of polynomials $\tau\widetilde{P}(s) + \nu\widetilde{Q}(s) + \widetilde{R}_0(s) + \sum_{i=1}^{\ell} a_i\widetilde{R}_i(s)$ under l_p-bounded uncertainties a_i. Application of the robust D-decomposition to the design of low-order controllers under uncertainties both in the plant and controller was demonstrated.

Construction of the boundaries of the domains of D-decomposition was shown to be reducible in this case to the solution of the fourth-order equations. The situations arising at occurrence of singularities were discussed in detail. Consideration was also given to the variant of uncertainty not only in the polynomial $\widetilde{R}(s)$, but also in the polynomials $\widetilde{P}(s)$ or $\widetilde{Q}(s)$.

For the complex parameter, an explicit formula to construct the boundaries of the domains of the robust D-decomposition was presented. Relationship between the robust D-decomposition and the circular arithmetics was demonstrated.

If there are uncertainties at least in two polynomials involved in the polynomial under consideration (in the complex case, for example)

$$\pi\left(\widetilde{D}_0(s) + \sum_{i=1}^{n} a_i \widetilde{D}_i(s)\right) + \widetilde{R}_0(s) + \sum_{i=1}^{m} b_i \widetilde{R}_i(s),$$
$$\|a\|_{p_1} \leqslant \gamma_a, \quad \|b\|_{p_2} \leqslant \gamma_b, \quad a \in \mathbb{C}^n, \quad b \in \mathbb{C}^m,$$

then in terms of the circular arithmetics solution of the basic equation of the robust D-decomposition $\Upsilon(\omega)$ is representable as

$$\Upsilon(\omega) = -C(R_0(\omega), \|R\|_{q_2}) C^{-1}(D_0(\omega), \|D\|_{q_1}),$$

where $q_1 = p_1/(p_1 - 1)$, $q_2 = p_2/(p_2 - 1)$. The quotient, as well as the product of two circular numbers, has a complex form and is a Pascal snail, rather than a circular number. The precise boundaries of the robust D-decomposition can be established from the two fourth-order systems, but it is more convenient to determine an approximate solution if the circular number $O(\omega) \supset \Upsilon(\omega)$ is taken at construction instead of $\Upsilon(\omega)$ (see, for example, [8]).

ACKNOWLEDGMENTS

The author would like to thank B.T. Polyak for his attention to the present paper and valuable criticism.

APPENDIX

Proof of Lemma 2. (1) It suffices to consider the case of $R_0 = 0, \gamma = 1$:

$$\mathcal{R} = \left\{ \sum_{i=1}^{\ell} a_i R_i = a^{\mathrm{T}} R : \|a\|_p \leqslant 1, a_i \in \mathbb{C}, R_i \in \mathbb{C} \right\}. \tag{A.1}$$

Let us take the complex number c which is equal in absolute value to unity and defines the direction on the plane and determine the distance from the zero to the boundary of the set (A.1) along this direction:

$$\max_{\|a\|_p \leqslant 1} \operatorname{Re}\left(c^* a^{\mathrm{T}} R\right) = \max_{\|a\|_p \leqslant 1} \operatorname{Re}\left(\left(c\,\overline{R}\right)^* a\right) = \|c\,\overline{R}\|_q = |c| \, \left\| \left|\overline{R}\right| \right\|_q = \|R\|_q, \tag{A.2}$$

where $q = p/(p-1)$, $a = (a_i)^{\mathrm{T}}$, $R = (R_i)^{\mathrm{T}}$. The asterisk denotes conjugation, and the overline, the complex conjugation. The derivation of (A.2) made use of the definition of the dual norm and the fact that an l_p-norm with a corresponding index will be dual to the l_p-norm [11]. The absolute value of the vector is understood as the elementwise value. This distance is independent of c; therefore, (A.1) defines a circle of radius $\|R\|_q$ centered at zero.

(2) Let us consider separately the real and imaginary parts of the set obeying (14). Obviously, this set is an ellipse because it is a linear map of the ℓ-dimensional sphere of the parameters $\mathcal{A}$

onto the two-dimensional space. Expression (16) in Lemma 2 is the notation of this ellipse using the nonnegative definite matrix $M = CC^{\mathrm{T}}$, where C is the map matrix [12]:

$$C = \begin{bmatrix} \operatorname{Re} R_1 & \cdots & \operatorname{Re} R_\ell \\ \operatorname{Im} R_1 & \cdots & \operatorname{Im} R_\ell \end{bmatrix}.$$

Proof of Theorem 2. As was shown in Section 2, the boundary of the domains of robust D-decomposition occurs in the union of the boundaries of the sets of solutions of the zero elimination equation and the degree reduction equation. As applied to the polynomial with complex parameter (21), the degree reduction equation similar to (8) for the real-parameter polynomial is as follows:

$$\pi \widetilde{d}_n + \widetilde{r}_{\mathcal{A}n} = 0, \tag{A.3}$$

where $\widetilde{r}_{\mathcal{A}n}$ is the set of values of the coefficients at s^n of the family of polynomials $\widetilde{R}_0(s) + \sum a_i \widetilde{R}_i(s)$, $a \in \mathcal{A}$. By Lemma 2, it is a circle of radius $\|\widetilde{r}_n\|_q$ centered at $\widetilde{r}_{0n}$, where $\widetilde{r}_n$ is the vector of the coefficients at s^n of the polynomials $\widetilde{R}_i$, $i = 1, \ldots, \ell$. The boundary of the solutions of Eq. (A.3) obeys the equation

$$\partial \Upsilon_s = \left\{ \pi : \left| \pi \widetilde{d}_n + \widetilde{r}_{0n} \right| = \|\widetilde{r}_n\|_q \right\}.$$

Now we set down the zero elimination condition for the polynomial with the complex parameter (counterpart of (10) for the real parameters) after passing to the variable ω:

$$\pi D(\omega) \in -R_0(\omega) - \sum_{i=1}^{\ell} a_i R_i(\omega), a_i \in \mathbb{C}, \|a\|_p \leqslant 1. \tag{A.4}$$

By Lemma 2, the set in the right-hand side of the inclusion is a circle. If $D(\omega) = 0$, then, depending on whether this circle includes the origin or not, the set of solutions of (A.4) $\Upsilon(\omega)$ will be the entire complex plane or the empty set. The first variant leads to $\Upsilon_\Omega = \mathbb{C}$ (all domains of the robust D-decomposition are empty and their boundary $\Upsilon = \{\emptyset\}$). In the second variant, $\Upsilon(\omega)$ makes no contribution to $\Upsilon_\Omega = \bigcup \Upsilon(\omega)$; therefore, these ω are skipped. If $D(\omega) \neq 0$, then the solutions of (A.4) make up a circle of the radius $\|R\|_q / |D(\omega)|$ centered at $-R_0(\omega)/D(\omega)$. Then, the boundary $\partial\Upsilon_\Omega$ is the envelope of the parametric-in-ω family of circles. Not all points of the envelope belong to the boundary $\partial\Upsilon_\Omega$ because of self-intersections. In its turn, the union of the boundaries does not coincide with the boundary Υ because of the intersections of the sets Υ_Ω and Υ_s. Both these remarks are reflected in the sign of inclusion in the expressions $\partial\Upsilon \subset \partial\Upsilon_\Omega \bigcap \partial\Upsilon_s$.

Let us pass to the notation with real variables $x_0(\omega) \doteq \operatorname{Re}(-R_0(\omega)/D(\omega))$, $y_0(\omega) \doteq \operatorname{Im}(-R_0(\omega)/D(\omega))$, $\rho(\omega) \doteq \|R(\omega)\|_q / |D(\omega)|$. Then, the points of the envelope are the solutions of the equation system (see [7])

$$\begin{cases} (x - x_0(\omega))^2 + (y - y_0(\omega))^2 - \rho(\omega)^2 = 0 \\ -2(x - x_0(\omega))x_0(\omega)' - 2(y - y_0(\omega))y_0(\omega)' - 2\rho(\omega)\rho(\omega)' = 0. \end{cases} \tag{A.5}$$

The first equation defines the circle; the second equation is its derivative with respect to the family parameter. By introducing the auxiliary variable ϕ, all solutions of the first system equation may be set down as $x = x_0(\omega) - \rho(\omega)\cos(\phi)$, $y = y_0(\omega) - \rho(\omega)\sin(\phi)$. By substituting ϕ in the second equation, we get $\cos(\phi)x_0(\omega)' + \sin(\phi)y_0(\omega)' = \rho(\omega)'$ if $\rho \neq 0$ or the identity if $\rho = 0$ (then, $x = x_0(\omega), y = y_0(\omega)$). Another auxiliary variable ψ such that

$$\begin{cases} \cos(\psi) = \dfrac{x_0(\omega)'}{\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2}} \\ \sin(\psi) = \dfrac{y_0(\omega)'}{\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2}} \end{cases}$$

gives rise to the equation

$$\cos(\phi - \psi) = \frac{\rho(\omega)'}{\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2}}.$$

A solution exists if only $\rho(\omega)'^2 \leqslant x_0(\omega)'^2 + y_0(\omega)'^2$. In the case where $x_0(\omega)' = y_0(\omega)' = \rho(\omega)' = 0$, the second equation of system (A.5) becomes an identity, and the envelope includes the circle or its part.

After simple trigonometrical developments, we get two solutions:

$$\begin{cases} \cos(\phi) = \dfrac{x_0(\omega)\rho(\omega)'}{x_0(\omega)'^2 + y_0(\omega)'^2} \mp \dfrac{y_0(\omega)\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2 - \rho(\omega)'^2}}{x_0(\omega)'^2 + y_0(\omega)'^2} \\ \sin(\phi) = \dfrac{y_0(\omega)\rho(\omega)'}{x_0(\omega)'^2 + y_0(\omega)'^2} \pm \dfrac{x_0(\omega)\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2 - \rho(\omega)'^2}}{x_0(\omega)'^2 + y_0(\omega)'^2} \end{cases}$$

and the final envelope equations (two branches):

$$\begin{cases} x = x_0(\omega) - \rho\dfrac{x_0(\omega)\rho(\omega)' \mp y_0(\omega)\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2 - \rho(\omega)'^2}}{x_0(\omega)'^2 + y_0(\omega)'^2} \\ y = y_0(\omega) - \rho\dfrac{y_0(\omega)\rho(\omega)' \pm x_0(\omega)\sqrt{x_0(\omega)'^2 + y_0(\omega)'^2 - \rho(\omega)'^2}}{x_0(\omega)'^2 + y_0(\omega)'^2}. \end{cases} \tag{A.6}$$

By returning to the complex coordinates, we obtain the equation of the circle of the family $\{\pi : |\pi - \pi_0(\omega)| = \rho(\omega)\}$, where $\pi \doteq x + jy$, $\pi_0(\omega) \doteq x_0(\omega) + jy_0(\omega)$, and envelope (A.6) is set down as

$$\pi = \pi_0(\omega) - \rho(\omega)\frac{\pi_0(\omega)'}{|\pi_0(\omega)'|}\left(\frac{\rho(\omega)'}{|\pi_0(\omega)'|} \pm j\sqrt{1 - \left(\frac{\rho(\omega)'}{|\pi_0(\omega)'|}\right)^2}\right).$$

This result coincides to within the notation of the variables with the expression for Υ_Ω from (23).

If π and a are the coefficients of the polynomial $\widetilde{G}(s)$ and consideration is given to the discrete case of $s = j\omega$, then $\rho(\omega)' = 0$ and the equation of the boundary of the D-decomposition becomes simpler:

$$\pi = \pi_0(\omega) \pm j\rho\frac{\pi_0(\omega)'}{|\pi_0(\omega)'|}.$$

REFERENCES

1. Neimark, Yu.I., *Ustoichivost' linearizovannykh sistem* (Stability of the Linearized Systems), Leningrad: LKVVIA, 1949.

2. Neimark, Yu.I., *Dinamicheskie sistemy i upravlyaemye protsessy* (Dynamic Systems and Controllable Processes), Moscow: Nauka, 1978.

3. Kharitonov, V.L., Asymptotic Stability of a Family of Systems of Linear Differential Equations, *Diff. Uravn.*, 1987, vol. 14, no. 11, pp. 2086–2088.

4. Polyak, B.T. and Tsypkin, Ya.Z., Frequency Criteria for Robust Stability and Aperiodicity of the Linear Systems, *Avtom. Telemekh.*, 1990, no. 9, pp. 45–54.

5. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.

6. Petrov, N.P. and Polyak, B.T., Robust D-decomposition, *Avtom. Telemekh.*, 1991, no. 11, pp. 41–53.

7. Luzin, N.N., *Differentsial'noe ischislenie* (Differential Calculus), Moscow: Vysshaya Shkola, 1960.

8. Polyak, B.T., Scherbakov, P.S., and Shmulyian, S.B., Construction of Value Set for Robustness Analysis via Circular Arithmetic, *Int. J. Robust Nonlinear Control*, 1994, vol. 4, pp. 371–385.

9. Kiselev, O.N. and Polyak, B.T., Design of Low-order Controllers by the H^∞ Criterion and the Maximum Robustness Criterion, *Avtom. Telemekh.*, 1999, no. 3, pp. 119–130.

10. Gryazina, E.N., On the Theory of D-decomposition, *Avtom. Telemekh.*, 2004, no. 12, pp. 15–28.

11. Horn, R. and Johnson, C., *Matrix Analysis*, New York: Cambridge Univ., 1985. Translated under the title *Matrichnyi analiz*, Moscow: Mir, 1989.

12. Schweppe, F., *Uncertain Dynamic Systems*, Englewood Cliffs: Prentice-Hall, 1973.

This paper was recommended for publication by B.T. Polyak, a member of the Editorial Board