ISSN 0012-2661, Differential Equations, 2007, Vol. 43, No. 3, pp. 346–356. © Pleiades Publishing, Ltd., 2007.
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ORDINARY DIFFERENTIAL EQUATIONS

On the Existence of a Smooth Solution of a Multidimensional Boundary Value Problem

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Received April 18, 2005

DOI: 10.1134/S0012266107030056

1. STATEMENT OF THE PROBLEM

Various considerations related to problems in control theory often lead to nonlinear boundary value problems of the form

$$\ddot{z}_k + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}^k(z)\dot{z}_i\dot{z}_j = 0, \qquad z_k(0) = 0, \qquad z_k(1) = 1, \qquad k = 1, \ldots, m, \tag{1}$$

with the additional condition

$$0 \le z_k(s) \le 1, \qquad 0 \le s \le 1, \qquad k = 1, \ldots, m, \tag{2}$$

where $a_{ij}^k(z)$, $i, j, k = 1, \ldots, m$, are smooth scalar functions.

Problem (1), (2) is a generalization of the multidimensional boundary value problem

$$\ddot{z}_j + \sum_{i=1}^{m} a_{ij}(z)\dot{z}_i\dot{z}_j = 0, \qquad z_j(0) = 0, \qquad z_j(1) = 1, \qquad j = 1, \ldots, m, \tag{3}$$

with the additional condition (2), where $a_{ij}(z)$, $i, j = 1, \ldots, m$, are smooth scalar functions. Sufficient conditions for the existence of a smooth solution of this problem were obtained in [1].

Problem (3), (2) arises in [2] when characterizing optimal control switching points as zeros of some functions for a class of control systems satisfying certain conditions. This problem also arises in [3] in the study of convexity conditions for the attainability set of a smooth control system linear in the phase variables.

In Section 2, we arrive at a boundary value problem of the form (1), (2) in the analysis of sufficient conditions for the convexity of an image of a convex polygon.

2. ON THE CONVEXITY OF AN IMAGE OF A CONVEX POLYGON

Theorems establishing conditions for the image of a convex set under a smooth mapping to be convex [4] have numerous applications in various control and optimization problems. For example, the two-point time optimization problem for the smooth nonlinear control system

$$x' = f(x, u), \qquad x \in \mathbb{R}^n, \qquad u(\cdot) \in \mathfrak{D}_U,$$

where $\mathfrak{D}_U$ is the set of bounded measurable functions ranging in a convex compact polygon $U \subset \mathbb{R}^m$, has a solution provided that the set $\mathfrak{F}(x) = \{f(x, u),\ u \in U\}$ is convex [5, pp. 203–204].

Let $U \subset \mathbb{R}^N$ be a convex compact polygon, and let $f : U \to \mathbb{R}^n$ be a smooth mapping of class C^2 satisfying the following condition: there exist smooth functionals $b_{ij}^k : U \to \mathbb{R}$ such that

$$\frac{\partial^2 f(u)}{\partial u^2}(w_i, w_j) = \sum_{k=1}^{m} b_{ij}^k(u) \frac{\partial f(u)}{\partial u} w_k, \tag{4}$$

where $w_1, \ldots, w_m$, $m \le N$, are linearly independent vectors tangent to edges $\Gamma_1, \ldots, \Gamma_m$ of the polygon U and such that $\operatorname{span}_{\mathbb{R}} \{w_1, \ldots, w_m\} = \operatorname{span}_{\mathbb{R}} U$.

Take arbitrary points $u, v \in U$. Since U is convex, it readily follows that there exist linearly independent vectors $\tilde{w}_1, \ldots, \tilde{w}_m$ tangent to edges $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_m$ of U and such that the m-dimensional parallelepiped Π with opposite vertices u and v and with edges parallel to $\tilde{w}_1, \ldots, \tilde{w}_m$ is contained in U.

Therefore, $f(U)$ is convex if there exist smooth scalar functions $\alpha_1(s), \ldots, \alpha_m(s)$ satisfying the inequalities

$$u_k \le \alpha_k(s) \le v_k, \qquad 0 \le s \le 1, \qquad k = 1, \ldots, m, \tag{5}$$

the boundary conditions

$$\alpha_k(0) = u_k, \qquad \alpha_k(1) = v_k, \qquad k = 1, \ldots, m,$$

and the relation

$$f\left(\sum_{j=1}^{m} \alpha_j(s) \tilde{w}_j\right) = f(u) + s(f(v) - f(u)), \tag{6}$$

where

$$u = \sum_{j=1}^{m} u_j \tilde{w}_j, \qquad v = \sum_{j=1}^{m} v_j \tilde{w}_j.$$

By differentiating relation (6) twice with respect to s, by introducing the notation

$$\alpha^* = \sum_{j=1}^{m} \alpha_j(s) \tilde{w}_j$$

for brevity, and by using (4), we obtain the chain of relations

$$\begin{aligned}
0 &= \frac{d^2}{ds^2} f\left(\sum_{j=1}^{m} \alpha_j \tilde{w}_j\right) = \frac{d}{ds}\left(\frac{\partial f(\alpha^*)}{\partial u} \sum_{j=1}^{m} \dot{\alpha}_j \tilde{w}_j\right) \\
&= \frac{\partial f(\alpha^*)}{\partial u} \sum_{k=1}^{m} \ddot{\alpha}_k \tilde{w}_k + \frac{\partial^2 f(\alpha^*)}{\partial u^2}\left(\sum_{i=1}^{m} \dot{\alpha}_i \tilde{w}_i, \sum_{j=1}^{m} \dot{\alpha}_j \tilde{w}_j\right) \\
&= \sum_{k=1}^{m} \ddot{\alpha}_k \frac{\partial f(\alpha^*)}{\partial u} \tilde{w}_k + \sum_{i=1}^{m} \sum_{j=1}^{m} \dot{\alpha}_i \dot{\alpha}_j \frac{\partial^2 f(\alpha^*)}{\partial u^2}(\tilde{w}_i, \tilde{w}_j) \\
&= \sum_{k=1}^{m} \ddot{\alpha}_k \frac{\partial f(\alpha^*)}{\partial u} \tilde{w}_k + \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} b_{ij}^k(\alpha^*) \dot{\alpha}_i \dot{\alpha}_j \frac{\partial f(\alpha^*)}{\partial u} \tilde{w}_k \\
&= \sum_{k=1}^{m} \left(\ddot{\alpha}_k + \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij}^k(\alpha^*) \dot{\alpha}_i \dot{\alpha}_j\right) \frac{\partial f(\alpha^*)}{\partial u} \tilde{w}_k,
\end{aligned}$$

which leads to the boundary value problem

$$\ddot{\alpha}_k + \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij}^k(\alpha) \dot{\alpha}_i \dot{\alpha}_j = 0, \qquad \alpha_k(0) = u_k, \qquad \alpha_k(1) = v_k, \quad k = 1, \ldots, m. \tag{7}$$

One can readily see that if $u_p = v_p$ for some p, then

$$\alpha_p(s) \equiv u_p = \mathrm{const}$$

and the dimension of system (7) can be reduced; therefore, we assume that $u_k \neq v_k$, $k = 1, \ldots, m$.

We set

$$\alpha_k(s) = u_k + (v_k - u_k)\, z_k(s), \qquad k = 1, \ldots, m;$$

then problem (7) acquires the form (1), where

$$a_{ij}^k = b_{ij}^k \frac{(v_i - u_i)\,(v_j - u_j)}{v_k - u_k}, \qquad i, j, k = 1, \ldots, m.$$

The additional conditions (5) acquire the form (2).

In Section 3, we derive sufficient conditions for the existence of a smooth solution of the boundary value problem (1) with the additional condition (2).

3. MAIN THEOREM

Theorem. *The boundary value problem* (1) *has a* C^2 *solution satisfying conditions* (2) *if the functions* $a_{ij}^k(z)$, $i, j, k = 1, \ldots, m$, *satisfy the system of differential equations*

$$\frac{\partial a_{ij}^k}{\partial z_p} = b_{ip}\left(a_{pj}^k + a_{jp}^k\right), \qquad k \neq p, \qquad i, j, k, p = 1, \ldots, m, \tag{8}$$

where $b_{ij}(z)$, $i, j = 1, \ldots, m$, *are smooth functions satisfying the system of differential equations*

$$\frac{\partial b_{ij}}{\partial z_p} = b_{ip} b_{pj}, \qquad j \neq p, \qquad i, j, p = 1, \ldots, m. \tag{9}$$

Proof. We seek a solution of problem (1) with conditions (2) as a solution of the boundary value problem

$$\ddot{z}_k + \sum_{i=1}^m b_{ik}(z) \dot{z}_i \dot{z}_k = 0, \qquad z_k(0) = 0, \qquad z_k(1) = 1, \qquad k = 1, \ldots, m, \tag{10}$$

with smooth functions $b_{ik}(z)$, $i, k = 1, \ldots, m$.

By [1], system (10) has a solution satisfying conditions (2) if

$$\frac{\partial b_{ik}}{\partial z_p} = b_{ip} b_{pk}, \qquad k \neq p, \qquad i, k, p = 1, \ldots, m.$$

In view of (10), system (1) acquires the form

$$-\sum_{i=1}^m b_{ik}(z) \dot{z}_i \dot{z}_k + \sum_{i=1}^m \sum_{j=1}^m a_{ij}^k(z) \dot{z}_i \dot{z}_j = 0, \qquad k = 1, \ldots, m,$$

or

$$\sum_{i=1}^m \left(\sum_{j=1}^m a_{ij}^k(z) \dot{z}_j - b_{ik}(z) \dot{z}_k \right) \dot{z}_i = 0, \qquad k = 1, \ldots, m.$$

The resulting relations hold for

$$\sum_{j=1}^m a_{ij}^k(z) \dot{z}_j = b_{ik}(z) \dot{z}_k, \qquad i, k = 1, \ldots, m. \tag{11}$$

As was shown in [1], there exist smooth functions $\lambda_k(z_k)$, $k = 1, \dots, m$, such that the solution of system (10) satisfies the system

$$\ddot{z}_k + \lambda_k(z_k)\dot{z}_k^2 = 0, \qquad z_k(0) = 0, \qquad z_k(1) = 1, \qquad k = 1, \dots, m; \tag{12}$$

moreover,

$$\lambda_k(z_k) = \sum_{i=1}^m b_{ik}(z)\dot{z}_i, \qquad k = 1, \dots, m. \tag{13}$$

Obviously, it follows from system (12) that

$$\dot{z}_k = C_{\lambda_k} \exp\left\{ -\int_0^{z_k} \lambda_k(\xi) d\xi \right\}, \qquad C_{\lambda_k} = \text{const} \neq 0, \qquad k = 1, \dots, m. \tag{14}$$

From (11) and (13), we have

$$\lambda_k(z_k) = \sum_{i=1}^m \sum_{j=1}^m a_{ij}^k(z) \frac{\dot{z}_i \dot{z}_j}{\dot{z}_k^2}, \qquad k = 1, \dots, m;$$

or, in view of (14) and the notation

$$\mu_k(z_k) = \int_0^{z_k} \lambda_k(\xi) d\xi, \qquad k = 1, \dots, m, \tag{15}$$

we obtain

$$\frac{\partial \mu_k}{\partial z_k} = \sum_{i=1}^m \sum_{j=1}^m \frac{C_{\lambda_i} C_{\lambda_j}}{C_{\lambda_k}^2} a_{ij}^k(z) \exp\left\{ 2\mu_k(z_k) - \mu_i(z_i) - \mu_j(z_j) \right\}, \qquad k = 1, \dots, m. \tag{16}$$

We arrive at the system of Pfaff equation with multidimensional time,

$$\frac{\partial x_i}{\partial t_j} = f_{ij}(t_1, \dots, t_n, x_1, \dots, x_m), \qquad i = 1, \dots, m, \qquad j = 1, \dots, n.$$

By [6, p. 175], a necessary and sufficient condition for the consistency of this system providing the existence and uniqueness of the solution for arbitrary initial conditions in the domain of the functions $f_{ij}(t, x)$ is given by the relations

$$\frac{\partial f_{ik}}{\partial t_j} + \sum_{p=1}^m \frac{\partial f_{ik}}{\partial x_p} f_{pj} \equiv \frac{\partial f_{ij}}{\partial t_k} + \sum_{p=1}^m \frac{\partial f_{ij}}{\partial x_p} f_{pk}, \qquad i = 1, \dots, m, \qquad k, j = 1, \dots, n. \tag{17}$$

For system (16), relations (17) become

$$\frac{\partial \gamma_k}{\partial z_p} + \frac{\partial \gamma_k}{\partial \mu_p} \gamma_p \equiv 0, \qquad k \neq p, \qquad k, p = 1, \dots, m, \tag{18}$$

where

$$\gamma_k = \sum_{i=1}^m \sum_{j=1}^m \frac{C_{\lambda_i} C_{\lambda_j}}{C_{\lambda_k}^2} a_{ij}^k(z) e^{2\mu_k - \mu_i - \mu_j}, \qquad k = 1, \dots, m.$$

Since

$$\frac{\partial \gamma_k}{\partial z_p} = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{C_{\lambda_i} C_{\lambda_j}}{C^2_{\lambda_k}} \frac{\partial a^k_{ij}(z)}{\partial z_p} e^{2\mu_k - \mu_i - \mu_j},$$

$$\frac{\partial \gamma_k}{\partial \mu_p} = -\sum_{i=1}^{m} \frac{C_{\lambda_i} C_{\lambda_p}}{C^2_{\lambda_k}} \left(a^k_{pi}(z) + a^k_{ip}(z)\right) e^{2\mu_k - \mu_p - \mu_i},$$

it follows that relations (18) become

$$\sum_{i=1}^{m}\sum_{j=1}^{m} C_{\lambda_i} C_{\lambda_j} \left(\frac{\partial a^k_{ij}}{\partial z_p} - \left(a^k_{pj} + a^k_{jp}\right) \sum_{s=1}^{m} a^p_{is} \frac{C_{\lambda_s}}{C_{\lambda_p}} e^{\mu_p - \mu_s} \right) e^{-\mu_i - \mu_j} \equiv 0,$$
$$k \neq p, \qquad k, p = 1, \ldots, m,$$

and hold for

$$\frac{\partial a^k_{ij}}{\partial z_p} = \left(a^k_{pj} + a^k_{jp}\right) \sum_{s=1}^{m} a^p_{is} \frac{C_{\lambda_s}}{C_{\lambda_p}} e^{\mu_p - \mu_s}, \qquad k \neq p, \qquad i, j, k, p = 1, \ldots, m. \tag{19}$$

By (11), (14), and (15), we have

$$\sum_{s=1}^{m} a^p_{is} \frac{C_{\lambda_s}}{C_{\lambda_p}} e^{\mu_p - \mu_s} = \sum_{s=1}^{m} a^p_{is} \frac{\dot{z}_s}{\dot{z}_p} = b_{ip}, \qquad i, p = 1, \ldots, m;$$

therefore, relation (19) becomes

$$\frac{\partial a^k_{ij}}{\partial z_p} = b_{ip} \left(a^k_{pj} + a^k_{jp}\right), \qquad k \neq p, \qquad i, j, k, p = 1, \ldots, m.$$

The proof of the theorem is complete.

4. EXAMPLES

The theorem implies that studying the existence of a smooth solution of the original multidimensional boundary value problem (1) essentially amounts to the analysis of systems (8) and (9) of differential equations. This analysis is not very difficult, and we carry it out for system (9) by way of example. In the two-dimensional case, the system will be solved completely, and in the general case, we indicate a sufficiently wide family of solutions.

First, we restrict our considerations to the two-dimensional case

$$\frac{\partial b_{11}}{\partial u_2} = b_{12} b_{21}, \qquad \frac{\partial b_{12}}{\partial u_1} = b_{11} b_{12}, \qquad \frac{\partial b_{21}}{\partial u_2} = b_{22} b_{21}, \qquad \frac{\partial b_{22}}{\partial u_1} = b_{21} b_{12}. \tag{20}$$

From the second and third equations of the system, we obtain

$$b_{12} = G(u_2) \exp\left\{ \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi \right\}, \qquad b_{21} = F(u_1) \exp\left\{ \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right\} \tag{21}$$

and substitute these expressions into the first and fourth equations:

$$\frac{\partial b_{ii}}{\partial u_{3-i}} = F(u_1)\, G(u_2) \exp\left\{ \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right\}, \qquad i = 1, 2,$$

or

$$\frac{\partial^2}{\partial u_1 \partial u_2} \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi = F(u_1)\, G(u_2) \exp\left\{ \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right\},$$

$$\frac{\partial^2}{\partial u_1 \partial u_2} \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta = F(u_1)\, G(u_2) \exp\left\{ \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right\}.$$

By adding and subtracting the resulting relations term by term, we obtain

$$\frac{\partial^2}{\partial u_1 \partial u_2} \left(\int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right) = 2F(u_1)\, G(u_2) \exp\left\{ \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right\} \tag{22}$$

and also

$$\frac{\partial^2}{\partial u_1 \partial u_2} \left(\int_0^{u_1} b_{11}(\xi, u_2)\, d\xi - \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta \right) = 0,$$

which implies that

$$\int_0^{u_1} b_{11}(\xi, u_2)\, d\xi - \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta = A(u_1) + B(u_2)\,. \tag{23}$$

Equation (22) has the form

$$w_{u_1 u_2} = 2F(u_1)\, G(u_2)\, e^w, \tag{24}$$

where

$$w = w(u_1, u_2) = \int_0^{u_1} b_{11}(\xi, u_2)\, d\xi + \int_0^{u_2} b_{22}(u_1, \eta)\, d\eta. \tag{25}$$

In Eq. (24), we pass to the new variables

$$\tilde{u}_1 = \int_0^{u_1} F(\xi) d\xi, \qquad \tilde{u}_2 = \int_0^{u_2} G(\eta) d\eta, \tag{26}$$

in which the equation becomes

$$w_{\tilde{u}_1 \tilde{u}_2} = 2e^w. \tag{27}$$

By [7, p. 174], the general solution of the differential equation

$$z_{xy} = a e^{\lambda z}, \qquad a, \lambda \in \mathbb{R},$$

has the form

$$z(x, y) = \frac{f(x) + g(y)}{\lambda} - \frac{2}{\lambda} \ln \left| k \int_0^x e^{f(\xi)} d\xi + \frac{a\lambda}{2k} \int_0^y e^{g(\eta)} d\eta \right|,$$

where $f(x)$ and $g(y)$ are arbitrary functions and $k \in \mathbb{R}$.

Consequently, Eq. (27) has the solution

$$w\left(\tilde{u}_1, \tilde{u}_2\right) = \varphi\left(\tilde{u}_1\right) + \psi\left(\tilde{u}_2\right) - 2\ln\left|k\int\limits_0^{\tilde{u}_1} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\tilde{u}_2} e^{\psi(\eta)} d\eta\right|,$$

where $\tilde{u}_1$ and $\tilde{u}_2$ are given by (26). Then relation (25) acquires the form

$$\int\limits_0^{u_1} b_{11}\left(\xi, u_2\right) d\xi + \int\limits_0^{u_2} b_{22}\left(u_1, \eta\right) d\eta$$
$$= \varphi\left(\int\limits_0^{u_1} F(\xi) d\xi\right) + \psi\left(\int\limits_0^{u_2} G(\eta) d\eta\right) - 2\ln\left|k\int\limits_0^{\int_0^{u_1} F(\xi)d\xi} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\int_0^{u_2} G(\eta)d\eta} e^{\psi(\eta)} d\eta\right|,$$

which, together with (23), implies that

$$\int\limits_0^{u_1} b_{11}\left(\xi, u_2\right) d\xi = \frac{A\left(u_1\right) + B\left(u_2\right)}{2} + \left[\varphi\left(\int\limits_0^{u_1} F(\xi) d\xi\right) + \psi\left(\int\limits_0^{u_2} G(\eta) d\eta\right)\right]\Big/ 2$$
$$- \ln\left|k\int\limits_0^{\int_0^{u_1} F(\xi)d\xi} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\int_0^{u_2} G(\eta)d\eta} e^{\psi(\eta)} d\eta\right|, \quad (28)$$

$$\int\limits_0^{u_2} b_{22}\left(u_1, \eta\right) d\eta = -\frac{A\left(u_1\right) + B\left(u_2\right)}{2} + \left[\varphi\left(\int\limits_0^{u_1} F(\xi) d\xi\right) + \psi\left(\int\limits_0^{u_2} G(\eta) d\eta\right)\right]\Big/ 2$$
$$- \ln\left|k\int\limits_0^{\int_0^{u_1} F(\xi)d\xi} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\int_0^{u_2} G(\eta)d\eta} e^{\psi(\eta)} d\eta\right|. \quad (29)$$

By differentiating relations (28) and (29) with respect to u_1 and u_2, respectively, we find b_{11} and b_{22}:

$$b_{11} = \frac{A'\left(u_1\right)}{2} + \varphi'\left(\int\limits_0^{u_1} F(\xi) d\xi\right) F\left(u_1\right)/2$$
$$- k\exp\left\{\varphi\left(\int\limits_0^{u_1} F(\xi) d\xi\right)\right\} F\left(u_1\right)\Big/\left(k\int\limits_0^{\int_0^{u_1} F(\xi)d\xi} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\int_0^{u_2} G(\eta)d\eta} e^{\psi(\eta)} d\eta\right), \quad (30)$$

$$b_{22} = -\frac{B'\left(u_2\right)}{2} + \psi'\left(\int\limits_0^{u_2} G(\eta) d\eta\right) G\left(u_2\right)/2$$
$$- \frac{1}{k}\exp\left\{\psi\left(\int\limits_0^{u_2} G(\eta) d\eta\right)\right\} G\left(u_2\right)\Big/\left(k\int\limits_0^{\int_0^{u_1} F(\xi)d\xi} e^{\varphi(\xi)} d\xi + \frac{1}{k}\int\limits_0^{\int_0^{u_2} G(\eta)d\eta} e^{\psi(\eta)} d\eta\right). \quad (31)$$

From (21), (28) and (29), we obtain

$$b_{12} = G(u_2)\exp\left\{\frac{A(u_1)+B(u_2)}{2}\right\}\exp\left\{\left(\varphi\left(\int_0^{u_1} F(\xi)d\xi\right)+\psi\left(\int_0^{u_2} G(\eta)d\eta\right)\right)\Big/2\right\}$$
$$\times\left(k\int_0^{\int_0^{u_1}F(\xi)d\xi} e^{\varphi(\xi)}d\xi+\frac{1}{k}\int_0^{\int_0^{u_2}G(\eta)d\eta} e^{\psi(\eta)}d\eta\right)^{-1}, \tag{32}$$

$$b_{21} = F(u_1)\exp\left\{-\frac{A(u_1)+B(u_2)}{2}\right\}\exp\left\{\left(\varphi\left(\int_0^{u_1} F(\xi)d\xi\right)+\psi\left(\int_0^{u_2} G(\eta)d\eta\right)\right)\Big/2\right\}$$
$$\times\left(k\int_0^{\int_0^{u_1}F(\xi)d\xi} e^{\varphi(\xi)}d\xi+\frac{1}{k}\int_0^{\int_0^{u_2}G(\eta)d\eta} e^{\psi(\eta)}d\eta\right)^{-1}. \tag{33}$$

Thus the solutions of system (9) are given by the functions (30)–(33), where A, B, F, G, φ, and ψ are arbitrary scalar functions and $k \in \mathbb{R}$.

In particular, if $b_{12} = 0$, then, obviously,

$$b_{11} = b_{11}(u_1), \qquad b_{12} = 0, \qquad b_{21} = F(u_1)\exp\left\{\int_0^{u_2} b_{22}(\eta)d\eta\right\}, \qquad b_{22} = b_{22}(u_2),$$

and if $b_{21} = 0$, then, in a similar way,

$$b_{11} = b_{11}(u_1), \qquad b_{12} = G(u_2)\exp\left\{\int_0^{u_1} b_{11}(\xi)d\xi\right\}, \qquad b_{21} = 0, \qquad b_{22} = b_{22}(u_2).$$

Now consider the multidimensional case. By setting $k = i$ in system (9), we obtain

$$\partial_i b_{ij} = b_{ii} b_{ij}, \qquad i \neq j,$$

or

$$\partial_i \ln|b_{ij}| = b_{ii}, \qquad i \neq j. \tag{34}$$

Further, by setting $k = j$ and $j = i$ in system (9), we obtain

$$\partial_j b_{ii} = b_{ij} b_{ji}, \qquad i \neq j. \tag{35}$$

By substituting relation (34) into the resulting equation, we obtain

$$\partial^2_{ij} \ln|b_{ij}| = b_{ij} b_{ji}, \qquad i \neq j. \tag{36}$$

The permutation of the indices i and j in this relation gives

$$\partial^2_{ij} \ln|b_{ji}| = b_{ij} b_{ji}, \qquad i \neq j. \tag{37}$$

The difference of relations (36) and (37) gives

$$\partial_{ij}^2 \ln \left|\frac{b_{ji}}{b_{ij}}\right| = \partial_{ij}^2 \ln \left|\frac{b_{ij}}{b_{ji}}\right| = 0, \qquad i \neq j, \tag{38}$$

and their sum gives

$$\partial_{ij}^2 \ln |b_{ji} b_{ij}| = 2 b_{ij} b_{ji}, \qquad i \neq j. \tag{39}$$

In particular, it follows from (38) that

$$\frac{b_{ji}}{b_{ij}} = F_j(\hat{u}_i)\, G_i(\hat{u}_j), \qquad i \neq j$$

(the symbol $\hat{u}_i$ indicates that u_i is not in present in the list of arguments); i.e.,

$$b_{ji} = b_{ij} F_j(\hat{u}_i)\, G_i(\hat{u}_j), \qquad i \neq j. \tag{40}$$

By substituting (40) into (39), we obtain the equation

$$\partial_{ij}^2 \ln |b_{ij}| = b_{ij}^2 F_j(\hat{u}_i)\, G_i(\hat{u}_j),$$

which, in particular, has the solution

$$b_{ij} = \left(\int_0^{u_j} F_j(\hat{u}_i)\, d\xi + \int_0^{u_i} G_i(\hat{u}_j)\, d\eta + R(\hat{u}_i, \hat{u}_j) \right)^{-1}, \qquad i \neq j. \tag{41}$$

Then it follows from (40) that

$$b_{ji} = F_j(\hat{u}_i)\, G_i(\hat{u}_j) \left(\int_0^{u_j} F_j(\hat{u}_i)\, d\xi + \int_0^{u_i} G_i(\hat{u}_j)\, d\eta + R(\hat{u}_i, \hat{u}_j) \right)^{-1}, \qquad i \neq j. \tag{42}$$

By (34), we have

$$b_{ii} = -G_i(\hat{u}_j) \left(\int_0^{u_j} F_j(\hat{u}_i)\, d\xi + \int_0^{u_i} G_i(\hat{u}_j)\, d\eta + R(\hat{u}_i, \hat{u}_j) \right)^{-1}; \tag{43}$$

therefore, from (35), we have

$$\begin{aligned} &- F_j(\hat{u}_i) \left(\int_0^{u_j} F_j(\hat{u}_i)\, d\xi + \int_0^{u_i} G_i(\hat{u}_j)\, d\eta + R(\hat{u}_i, \hat{u}_j) \right)^{-2} \\ &\quad = F_j(\hat{u}_i)\, G_i(\hat{u}_j) \left(\int_0^{u_j} F_j(\hat{u}_i)\, d\xi + \int_0^{u_i} G_i(\hat{u}_j)\, d\eta + R(\hat{u}_i, \hat{u}_j) \right)^{-2}, \qquad i \neq j, \end{aligned}$$

which implies the identity

$$G_i(\hat{u}_j) \equiv -1. \tag{44}$$

By (44), relations (41)–(43) acquire the form

$$b_{ii} = \left(-u_i + \int_0^{u_j} F_j(\hat{u}_i)\, d\xi + R(\hat{u}_i, \hat{u}_j) \right)^{-1}, \tag{45}$$

$$b_{ij} = \left(-u_i + \int_0^{u_j} F_j(\hat{u}_i)\, d\xi + R(\hat{u}_i, \hat{u}_j) \right)^{-1}, \qquad i \neq j, \tag{46}$$

$$b_{ji} = -F_j(\hat{u}_i) \left(-u_i + \int_0^{u_j} F_j(\hat{u}_i)\, d\xi + R(\hat{u}_i, \hat{u}_j) \right)^{-1}, \qquad i \neq j. \tag{47}$$

By exchanging the indices i and j in (46) and by comparing the resulting relation with (47), we obtain

$$-u_i + \int_0^{u_j} F_j(\hat{u}_i)\, d\xi + R(\hat{u}_i, \hat{u}_j) = -F_j(\hat{u}_i) \left(-u_j + \int_0^{u_i} F_i(\hat{u}_j)\, d\xi + R(\hat{u}_i, \hat{u}_j) \right), \qquad i \neq j.$$

After the differentiation with respect to u_i, the last relation implies that

$$F_j(\hat{u}_i) = c_j = \text{const}, \qquad i \neq j;$$

i.e.,

$$b_{ii} = \left(-u_i + c_j u_j + R(\hat{u}_i, \hat{u}_j) \right)^{-1}.$$

Since the index j on the right-hand side is arbitrary, we have

$$b_{ii} = \left(-u_i + \sum_{k \neq i} c_k u_k + \text{const} \right)^{-1}.$$

It remains to note that $b_{ij} = b_{ii}$ by (46) and (45) and $b_{ji} = -c_j b_{ii}$ by (46) and (47). Finally,

$$b_{ij} = b_i = \left(-u_i + \sum_{k \neq i} c_k u_k + c_0 \right)^{-1},$$

or

$$b_{ij} = b_i = -c_i \left(\sum_k c_k u_k + c_0 \right)^{-1}.$$

Some properties of systems (8) and (9) were considered in [2] (see also [8]).

5. CONCLUDING REMARKS

Let us introduce the connection determined by the Christoffel symbols

$$\Gamma_{ij}^k = a_{ik}^j(z). \tag{48}$$

For the connection (48), the torsion tensor $\Gamma_{[ij]}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ has the form $\Gamma_{[ij]}^k = a_{ik}^j - a_{jk}^i$.

The Riemann curvature tensor [9, p. 362] given by the relation

$$-R^k_{q;ij} = \frac{\partial}{\partial z^i}\Gamma^k_{qj} - \frac{\partial}{\partial z^j}\Gamma^k_{qi} + \Gamma^p_{qj}\Gamma^k_{pi} - \Gamma^p_{qi}\Gamma^k_{pj}$$

(where p is a summation index) acquires the form

$$\begin{aligned} -R^k_{q;ij} &= \frac{\partial}{\partial z^i}a^j_{qk} - \frac{\partial}{\partial z^j}a^i_{qk} + a^j_{qp}a^i_{pk} - a^i_{qp}a^j_{pk} \\ &= \begin{cases} b_{qi}\left(a^j_{ik} + a^j_{ki}\right) - b_{qj}\left(a^i_{jk} + a^i_{kj}\right) + a^j_{qp}a^i_{pk} - a^i_{qp}a^j_{pk} & \text{for} \quad i \neq j \\ 0 & \text{for} \quad i = j. \end{cases} \end{aligned}$$

Therefore, the connection (48) is flat if

$$b_{qi}\left(a^j_{ik} + a^j_{ki}\right) = \sum_{p=1}^{m} a^i_{qp}a^j_{pk}, \qquad i \neq j, \qquad i, j, k, q = 1, \ldots, m.$$

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