ISSN 0005-1179, Automation and Remote Control, 2006, Vol. 67, No. 6, pp. 933–944. © Pleiades Publishing, Inc., 2006.
Original Russian Text © M.V. Topunov, 2006, published in Avtomatika i Telemekhanika, 2006, No. 6, pp. 113–125.

DETERMINATE SYSTEMS

A Class of μ-commutative Bilinear Control Systems

M. V. Topunov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Received December 5, 2005

Abstract—A class of μ-commutative bilinear control systems is defined and the properties of coefficients of a μ-commutative system are studied. A μ-commutative scalar control system is studied and an explicit representation for its input-output mapping is derived. This mapping is used to study the properties of a system, particularly the geometry of its reachable set.

PACS number: 07.05.Dz, 02.30.Yy, 02.30.Hq

DOI: 10.1134/S0005117906060087

1. INTRODUCTION

Let us consider the smooth control system

$$\dot{x} = f(x, u), \quad x \in \mathcal{M}, \quad u(\cdot) \in \mathfrak{D}_U, \tag{1}$$

where $\mathcal{M}$ is a smooth C^∞-manifold regularly imbedded in $\mathbb{R}^n$, $\mathfrak{D}_U$ the admissible control set of all bounded measurable functions of time t taking values in a convex compact polyhedron $U \subset \mathbb{R}^m$, and $f : \mathcal{M} \times U \to \mathbb{R}^n$ is a smooth mapping (of the class C^2).

In a broad sense (see [1]), the control problem is reduced to studying the properties of the *input-output mapping*

$$u(\cdot) \mapsto F_{x_0,T}(u), \tag{2}$$

where $F_{x_0,T}(u)$ is the point of the phase space to which the control system (1) passes from the initial point $x_0 \in \mathcal{M}$ in time $T > 0$ under the action of an admissible control $u(\cdot) \in \mathfrak{D}_U$. A main difficulty in studying nonlinear control systems (1) is that, unlike for linear systems, an explicit representation for the input-output mapping for nonlinear systems cannot be easily found.

The class that is next in complexity to linear systems is formed by *smooth control systems of constant rank* [2, 3]. Precisely they are close to linear control systems. System (1) is said to a smooth control system of constant rank if, for any point $x_0 \in \mathcal{M}$ and any $T > 0$, the rank of mapping (2) does not depend on $u(\cdot) \in \mathfrak{D}_U$ (but may depend on $x_0 \in \mathcal{M}$ and $T > 0$).

Sufficient conditions under which system (1) is of constant rank, the so-called bang-band principle, are formulated in [2]. System (1) satisfies the *bang-bang principle* if, for any point $(x_0, u_0) \in \mathcal{M} \times \mathbb{R}^m$, there exists a neighborhood V such that $(x_0, u_0) \in V$ and the bilinear mappings

$$a^\alpha_\beta(x, u; \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m,$$

smoothly dependent on x and u such that

$$\frac{\partial}{\partial u}\mathrm{ad}^\alpha f(x,u)\frac{\partial f(x,u)}{\partial u}(v,w) = \sum_{\beta=0}^{\alpha}\left(\left(a^\alpha_\beta(x,u;v,w)\right)^*\left(\mathrm{ad}^\beta f(x,u)\frac{\partial f(x,u)}{\partial u}\right)^*\right)^*, \quad \alpha = 0, 1, \ldots,$$

for all $(x, u) \in V$ and $v, w \in \mathbb{R}^m$, where $\mathrm{ad} f g = [f, g]$ is a commutator of smooth vector fields and a^* is an element of the adjoint of the space of vectors a.

Smooth control systems are investigated in [4–7]. The geometry of the reachable set of a quasi-commutative bilinear is studied in [4] and its results are generalized to the class of bilinear constant-rank control systems in [5] and to the class of smooth control systems that are linear in phase variables in [6]. Sufficient conditions for the embeddability of the reachable set of smooth constant-rank control systems that are linear in phase variables are formulated in [7] . Related problems of control systems are described in [8–12].

In this paper, we introduce and study a class of smooth constant-rank control systems different from quasi-commutative systems, viz., μ-commutative bilinear systems. We shall investigate the properties of the coefficients of a μ-commutative system and derive an explicit representation for the input-output mapping of a μ-commutative scalar control system. These properties are helpful in studying the properties of the system, particularly the geometry of its reachable set.

2. FORMULATION OF THE PROBLEM

Let us consider the bilinear control system

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i B_i \right) x, \quad x \in \mathbb{R}^n, \quad u(\cdot) = (u_1, \ldots, u_m) \in \mathfrak{D}_U, \tag{3}$$

where A and B_i, $i = 1, \ldots, m$, are constant $n \times n$ matrices and $\mathfrak{D}_U$ is the admissible control set of all bounded measurable vector functions of time t with values in a convex compact polyhedron $U \subset \mathbb{R}^m$.

The bang-bang principle for system (3), according to [3], takes the form

$$[B_i, \mathrm{ad}^k A B_j] = \sum_{\alpha=0}^{k} \sum_{\beta=1}^{m} a_{\alpha\beta}^{ijk} \mathrm{ad}^{\alpha} A B_{\beta}, \quad a_{\alpha\beta}^{ijk} \in \mathbb{R}, \quad i, j = 1, \ldots, m, \quad k = 0, 1, \ldots, \tag{4}$$

where

$$\begin{gathered} \mathrm{ad}^0 A B = B, \\ \mathrm{ad} A B = [A, B] = AB - BA, \\ \mathrm{ad}^{k+1} A B = [A, \mathrm{ad}^k A B], \quad k = 1, 2, \ldots . \end{gathered}$$

The simplest class of smooth constant-rank control systems is the class of *commutative systems*. A bilinear control system (3) is said to be commutative if it satisfies the *commutativity conditions*

$$[A, B_i] = 0, \quad [B_i, B_j] = 0, \quad i, j = 1, \ldots, m.$$

Obviously, its input-output mapping can be represented in the form

$$F_{x_0,T}(u) = e^{TA} \prod_{i=1}^{m} e^{B_i \int\limits_0^T u_i(\theta)\, d\theta} x_0, \quad u(\cdot) = (u_1, \ldots, u_m) \in \mathfrak{D}_U.$$

A far more complicated class of smooth constant-rank control systems is the class of *quasi-commutative systems*. A bilinear control system (3) is said to be quasi-commutative if it satisfies the *quasi-commutativity conditions*

$$[B_i, \mathrm{ad}^k A B_j] = 0, \quad i, j = 1, \ldots, m, \quad k = 0, 1, \ldots .$$

According to [13, 4], its input-output mapping can be represented as

$$F_{x_0,T}(u) = e^{TA} e^{\int_0^T e^{-\theta \operatorname{ad} A} \sum_{i=1}^m B_i u_i(\theta)\, d\theta} x_0, \quad u(\cdot) = (u_1, \ldots, u_m) \in \mathfrak{D}_U.$$

A bilinear control system (3) is said to be *μ-commutative* if it satisfies the *μ-commutative conditions*

$$\operatorname{ad}^\alpha AB_i \operatorname{ad}^\beta AB_j = \sum_{k=1}^m \mu_{ijk}^{\alpha\beta} \operatorname{ad}^{\alpha+\beta} AB_k, \quad \mu_{ijk}^{\alpha\beta} \in \mathbb{R}, \quad i,j = 1, \ldots, m, \quad \alpha, \beta = 0, 1, \ldots . \tag{5}$$

As will be shown below, the μ-commutative conditions (5) are obviously encountered in studying the input-output mapping of system (3).

Theorem 1. *The μ-commutative system* (3), (5) *is a control system of constant rank.*

The proof of Theorem 1 is given in the Appendix.

3. PROPERTIES OF THE COEFFICIENTS OF A μ-COMMUTATIVE SYSTEM

The coefficients $\mu_{ijk}^{\alpha\beta}$, $\alpha, \beta = 0, 1, \ldots,$ $i, j, k = 1, 2, \ldots, m$, in relations (5) are called the *μ-coefficients* of system (3), (5) (or *coefficients of the μ-commutative system* (3), (5)). They have certain important properties.

Theorem 2. *If*

$$\operatorname{rank}\{\operatorname{ad}^{\alpha+\beta+1} AB_1, \operatorname{ad}^{\alpha+\beta+1} AB_2, \ldots, \operatorname{ad}^{\alpha+\beta+1} AB_m\} = m,$$

then

$$\mu_{ijk}^{\alpha\beta} = \mu_{ijk}^{\alpha+1,\beta} + \mu_{ijk}^{\alpha,\beta+1}, \quad \alpha, \beta = 0, 1, \ldots, \quad i, j, k = 1, \ldots, m.$$

Theorem 3. *If*

$$\operatorname{rank}\{\operatorname{ad}^p AB_1, \operatorname{ad}^p AB_2, \ldots, \operatorname{ad}^p AB_m\} = m,$$

then

$$\mu_{ijk}^{00} = \sum_{q=0}^p C_p^q \mu_{ijk}^{q,p-q}, \quad p = 0, 1, \ldots, \quad i, j, k = 1, \ldots, m,$$

where C_p^q are binomial coefficients.

Theorem 4. *If*

$$\operatorname{rank}\{\operatorname{ad}^{\alpha+\beta+\gamma} AB_1, \operatorname{ad}^{\alpha+\beta+\gamma} AB_2, \ldots, \operatorname{ad}^{\alpha+\beta+\gamma} AB_m\} = m,$$

then

$$\sum_{k=1}^m \mu_{ijk}^{\alpha\beta} \mu_{kpq}^{\alpha+\beta,\gamma} = \sum_{k=1}^m \mu_{ikq}^{\alpha,\beta+\gamma} \mu_{jpk}^{\beta\gamma}, \quad \alpha, \beta, \gamma = 0, 1, \ldots, \quad i, j, p, q = 1, \ldots, m.$$

The proofs of Theorems 2–4 are given in the Appendix.

According to these theorem, the internal relations between μ-coefficients of a bilinear system in general are rather complicated. Nevertheless, in the next section, we shall derive a direct representation for the "structure" of a system with scalar control.

4. COEFFICIENTS OF A μ-COMMUTATIVE SYSTEM WITH SCALAR CONTROL

Let us consider a μ-commutative system with scalar control

$$\dot{x} = (A + uB)x, \quad x \in \mathbb{R}^n, \quad u(\cdot) \in \mathfrak{D}_U, \tag{6}$$

where A and B are constant $n \times n$ matrices and $\mathfrak{D}_U$ is the admissible control set of all bounded measurable scalar functions of time t.

Since the conditions for the μ-commutativity of system (6) are

$$\mathrm{ad}^\alpha AB\mathrm{ad}^\beta AB = \mu^{\alpha\beta}\mathrm{ad}^{\alpha+\beta}AB, \quad \mu^{\alpha\beta} \in \mathbb{R}, \quad \alpha, \beta = 0, 1, \dots . \tag{7}$$

Theorems 2–4 take the following form.

Theorem $2'$**.** *If* $\mathrm{ad}^{\alpha+\beta+1}AB \neq 0$*, then*

$$\mu^{\alpha\beta} = \mu^{\alpha+1,\beta} + \mu^{\alpha,\beta+1}, \quad \alpha, \beta = 0, 1, \dots .$$

Theorem $3'$**.** *If* $\mathrm{ad}^p AB \neq 0$*, then*

$$\mu^{00} = \sum_{q=0}^{p} C_p^q \mu^{q,p-q}, \quad p = 0, 1, \dots ,$$

where C_p^q *are binomial coefficients.*

Theorem $4'$**.** *If* $\mathrm{ad}^{\alpha+\beta+\gamma}AB \neq 0$*, then*

$$\mu^{\alpha\beta}\mu^{\alpha+\beta,\gamma} = \mu^{\alpha,\beta+\gamma}\mu^{\beta\gamma}, \quad \alpha, \beta, \gamma = 0, 1, \dots .$$

The last theorem implies the following obvious corollaries.

Corollary 1. *If* $\mathrm{ad}^{\alpha+\beta}AB \neq 0$*, then*

$$\begin{aligned}
\mu^{\alpha\beta}(\mu^{0\alpha} - \mu^{0,\alpha+\beta}) &= 0, \quad \alpha, \beta = 0, 1, \dots , \\
\mu^{\alpha\beta}(\mu^{\alpha+\beta,0} - \mu^{\beta 0}) &= 0, \quad \alpha, \beta = 0, 1, \dots , \\
\mu^{\alpha\beta}(\mu^{\alpha 0} - \mu^{0\beta}) &= 0, \quad \alpha, \beta = 0, 1, \dots .
\end{aligned}$$

Corollary 2. *If* $\mathrm{ad}^\alpha AB \neq 0$*, then*

$$\begin{aligned}
\mu^{0\alpha}(\mu^{00} - \mu^{0\alpha}) &= 0, \quad \alpha = 0, 1, \dots , \\
\mu^{\alpha 0}(\mu^{\alpha 0} - \mu^{00}) &= 0, \quad \alpha = 0, 1, \dots .
\end{aligned}$$

These results directly imply that the class of μ-commutative bilinear systems with scalar control decomposes into a class of *left* μ-commutative bilinear systems satisfying the condition

$$\mu^{\alpha\beta} = \begin{cases} \mu^{00}, & \beta = 0 \\ 0, & \beta \neq 0, \end{cases} \quad \alpha, \beta = 0, 1, \dots , \tag{8}$$

and a class of *right* μ-commutative bilinear systems satisfying the condition

$$\mu^{\alpha\beta} = \begin{cases} \mu^{00}, & \alpha = 0 \\ 0, & \alpha \neq 0, \end{cases} \quad \alpha, \beta = 0, 1, \dots . \tag{9}$$

It is easily seen that if $\mu^{00} = 0$, then a μ-commutative system with scalar control degenerates to a quasi-commutative system.

If the μ-coefficients are represented as a triangular table

$$\begin{array}{ccccccc} & & & \mu^{00} & & & \\ & & \mu^{10} & & \mu^{01} & & \\ & \mu^{20} & & \mu^{11} & & \mu^{02} & \\ \mu^{30} & & \mu^{21} & & \mu^{12} & & \mu^{03} \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ,} \end{array}$$

then this table takes the form

$$\begin{array}{ccccccc} & & & \mu^{00} & & & \\ & & \mu^{00} & & 0 & & \\ & \mu^{00} & & 0 & & 0 & \\ \mu^{00} & & 0 & & 0 & & 0 \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots ,} \end{array}$$

for "left" systems, and the form

$$\begin{array}{ccccccc} & & & \mu^{00} & & & \\ & & 0 & & \mu^{00} & & \\ & 0 & & 0 & & \mu^{00} & \\ 0 & & 0 & & 0 & & \mu^{00} \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots ,} \end{array}$$

for "right" systems.

If

$$\widetilde{\alpha} = \min\{\alpha, \quad \mathrm{ad}^{\alpha} AB \neq 0\},$$

then these tables contain $(\widetilde{\alpha} + 1)$ nonzero rows.

5. FUNDAMENTAL MATRIX OF A μ-COMMUTATIVE SCALAR CONTROL SYSTEM

Let $\Omega_0^t(P)$ be the *fundamental matrix* (matrizant) of the matrix differential equation

$$\frac{dX}{dt} = P(t)X$$

with initial condition $X(0) = E$, where E is a unit matrix.

According to [3, 14],

$$\Omega_0^t(A) = E + \int\limits_0^t A(\tau_1)\, d\tau_1 + \int\limits_0^t A(\tau_1)\, d\tau_1 \int\limits_0^{\tau_1} A(\tau_2)\, d\tau_2 + \dots, \tag{10}$$

$$\Omega_0^t(A) = e^{At}, \quad A = \mathrm{const}, \tag{11}$$

$$\Omega_0^t(A + B) = \Omega_0^t(A)\Omega_0^t(\mathrm{Ad}\Omega_0^{\theta}(A)B), \tag{12}$$

where

$$\mathrm{Ad}AB = A^{-1}BA.$$

The input-output mapping for system (3) can obviously be represented as

$$F_{x_0,t}(u) = \Omega_0^t \left(A + \sum_{i=1}^{m} B_i u_i \right) x_0, \quad u(\cdot) = (u_1, \ldots, u_m) \in \mathfrak{D}_U,$$

and for a bilinear scalar control system (6) as

$$F_{x_0,t}(u) = \Omega_0^t (A + Bu) x_0, \quad u(\cdot) \in \mathfrak{D}_U.$$

Let us study the structure of the fundamental matrix $\Omega_0^t(A + Bu)$ of system (6). According to (11) and (12),

$$\begin{aligned}\Omega_0^t(A + Bu) &= \Omega_0^t(A)\Omega_0^t(\mathrm{Ad}\Omega_0^\theta(A)Bu) = e^{tA}\Omega_0^t(\mathrm{Ad}e^{\theta A}Bu) = e^{tA}\Omega_0^t(e^{-\theta \mathrm{ad} A}Bu) \\ &= e^{tA}\Omega_0^t\left(\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB\theta^p u\right) = e^{tA}\Omega_0^t(Q(\theta)),\end{aligned}$$

where

$$Q(\theta) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB\nu_p^{(0)}(\theta), \quad \nu_p^{(0)}(\theta) = \theta^p u, \quad p = 0, 1, \ldots .$$

Then

$$\begin{aligned}Q(\theta)\int_0^\theta Q(\xi)\,d\xi &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB\nu_p^{(0)}(\theta) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB \int_0^\theta \nu_p^{(0)}(\xi)\,d\xi \\ &= \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{(-1)^k}{k!} \mathrm{ad}^k AB\nu_k^{(0)}(\theta) \frac{(-1)^{p-k}}{(p-k)!} \mathrm{ad}^{p-k} AB \int_0^\theta \nu_{p-k}^{(0)}(\xi)\,d\xi \\ &= \sum_{p=0}^{\infty}(-1)^p \sum_{k=0}^{p} \frac{1}{k!(p-k)!} \mathrm{ad}^k AB\mathrm{ad}^{p-k} AB\nu_k^{(0)}(\theta) \int_0^\theta \nu_{p-k}^{(0)}(\xi)\,d\xi \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB \sum_{k=0}^{p} C_p^k \mu^{k,p-k} \nu_k^{(0)}(\theta) \int_0^\theta \nu_{p-k}^{(0)}(\xi)\,d\xi = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \mathrm{ad}^p AB\nu_p^{(1)}(\theta),\end{aligned}$$

where

$$\nu_p^{(1)}(\theta) = \sum_{k=0}^{p} C_p^k \mu^{k,p-k} \nu_k^{(0)}(\theta) \int_0^\theta \nu_{p-k}^{(0)}(\xi)\,d\xi, \quad p = 0, 1, \ldots .$$

Similarly, we obtain

$$Q(\theta)\int\limits_0^\theta Q(\xi)\,d\xi\int\limits_0^\xi Q(\eta)\,d\eta=\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\nu_p^{(0)}(\theta)\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\int\limits_0^\theta\nu_p^{(1)}(\xi)\,d\xi$$

$$=\sum_{p=0}^\infty\sum_{k=0}^p\frac{(-1)^k}{k!}\mathrm{ad}^kAB\nu_k^{(0)}(\theta)\frac{(-1)^{p-k}}{(p-k)!}\mathrm{ad}^{p-k}AB\int\limits_0^\theta\nu_{p-k}^{(1)}(\xi)\,d\xi$$

$$=\sum_{p=0}^\infty(-1)^p\sum_{k=0}^p\frac{1}{k!(p-k)!}\mathrm{ad}^kAB\mathrm{ad}^{p-k}AB\nu_k^{(0)}(\theta)\int\limits_0^\theta\nu_{p-k}^{(1)}(\xi)\,d\xi$$

$$=\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta\nu_{p-k}^{(1)}(\xi)\,d\xi=\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\nu_p^{(2)}(\theta),$$

where

$$\nu_p^{(2)}(\theta)=\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta\nu_{p-k}^{(1)}(\xi)\,d\xi,\quad p=0,1,\ldots,$$

etc. Hence, according to (10), we obtain

$$\Omega_0^t(Q(\theta))=E+\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\int\limits_0^t\nu_p^{(0)}(\theta)\,d\theta+\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\int\limits_0^t\nu_p^{(1)}(\theta)\,d\theta+\ldots$$

$$=E+\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\int\limits_0^t\sum_{s=0}^\infty\nu_p^{(s)}(\theta)\,d\theta=E+\sum_{p=0}^\infty\frac{(-1)^p}{p!}\mathrm{ad}^pAB\int\limits_0^tM_p(\theta)\,d\theta,$$

where

$$M_p(\theta)=\sum_{s=0}^\infty\nu_p^{(s)}(\theta),\quad p=0,1,\ldots,$$

and

$$\nu_p^{(s)}(\theta)=\begin{cases}\theta^pu, & s=0\\ \sum\limits_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta\nu_{p-k}^{(s-1)}(\xi)\,d\xi, & s=1,2,\ldots.\end{cases}$$

The functions $M_p(\theta)$, $p=0,1,\ldots$, are called the *components* of the fundamental matrix of system (6). Note that

$$M_p(\theta)=\sum_{s=0}^\infty\nu_p^{(s)}(\theta)=\nu_p^{(0)}(\theta)+\sum_{s=1}^\infty\nu_p^{(s)}(\theta)=\theta^pu+\sum_{s=1}^\infty\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta\nu_{p-k}^{(s-1)}(\xi)\,d\xi$$

$$=\theta^pu+\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta\sum_{s=1}^\infty\nu_{p-k}^{(s-1)}(\xi)\,d\xi=\theta^pu+\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta M_{p-k}(\xi)\,d\xi,$$

i.e.,

$$M_p(\theta)=\theta^pu+\sum_{k=0}^pC_p^k\mu^{k,p-k}\nu_k^{(0)}(\theta)\int\limits_0^\theta M_{p-k}(\xi)\,d\xi,\quad p=0,1,\ldots.\tag{13}$$

Hence

$$M_0(\theta) = u\left(1 + \mu^{00}\int\limits_0^{\theta} M_0(\xi)\,d\xi\right).$$

Consequently,

$$M_0(\theta) = ue^{\mu^{00}\int\limits_0^{\theta} u(\xi)d\xi}. \tag{14}$$

For a “left” system (6), according to (8), from (13) we obtain

$$M_p(\theta) = \theta^p u\left(1 + \mu^{00}\int\limits_0^{\theta} M_0(\xi)\,d\xi\right), \quad p = 0, 1, \ldots,$$

or, by virtue of (14),

$$M_p(\theta) = \theta^p ue^{\mu^{00}\int\limits_0^{\theta} u(\xi)d\xi}, \quad p = 0, 1, \ldots. \tag{15}$$

For a “right” system (6), according to (9), from (13) we obtain

$$M_p(\theta) = u\left(\theta^p + \mu^{00}\int\limits_0^{\theta} M_p(\xi)\,d\xi\right), \quad p = 0, 1, \ldots.$$

Hence

$$M_p(\theta) = \begin{cases} ue^{\mu^{00}\int\limits_0^{\theta} u(\xi)d\xi}, & p = 0 \\ ue^{\mu^{00}\int\limits_0^{\theta} u(\xi)d\xi}\int\limits_0^{\theta}(\xi^p)'e^{-\mu^{00}\int\limits_0^{\xi} u(\eta)d\eta}d\xi, & p = 1, 2\ldots. \end{cases} \tag{16}$$

6. AN EXAMPLE

Let us consider a two-dimensional bilinear scalar-control system

$$\begin{cases} x_1' = cx_1 + aux_2 \\ x_2' = (d + bu)x_2, \end{cases} \quad |u| \leqslant 1, \tag{17}$$

where

$$a, b, c, d = \text{const}, \quad a, b \neq 0, \quad c \neq d.$$

Moreover,

$$A = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}.$$

It is easily seen that

$$B^2 = \begin{pmatrix} 0 & ab \\ 0 & b^2 \end{pmatrix} = bB,$$

$$\mathrm{ad}^\alpha AB = \begin{pmatrix} 0 & a(c-d)^\alpha \\ 0 & 0 \end{pmatrix}, \quad \alpha = 1, 2, \dots,$$

$$B\mathrm{ad}^\alpha AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha = 1, 2, \dots,$$

$$(\mathrm{ad}^\alpha AB)B = \begin{pmatrix} 0 & ab(c-d)^\alpha \\ 0 & 0 \end{pmatrix} = b\mathrm{ad}^\alpha AB, \quad \alpha = 1, 2, \dots,$$

$$\mathrm{ad}^\alpha AB\mathrm{ad}^\beta AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha, \beta = 1, 2, \dots,$$

$$[B, \mathrm{ad}^\alpha AB] = \begin{pmatrix} 0 & -ab(c-d)^\alpha \\ 0 & 0 \end{pmatrix}, \quad \alpha = 1, 2, \dots .$$

Hence system (17) is neither quasi-commutative, nor commutative. It is a "left" μ-commutative system with coefficients

$$\mu^{\alpha\beta} = \begin{cases} b, & \beta = 0 \\ 0, & \beta \neq 0, \end{cases} \qquad \alpha, \beta = 0, 1, \dots .$$

Then, according to (15), the components $M_p(\theta)$ of the fundamental matrix of system (17) are

$$M_p(\theta) = \theta^p u e^{b\int\limits_0^\theta u(\xi)d\xi}, \quad p = 0, 1, \dots .$$

7. CONCLUSIONS

A class of μ-commutative bilinear systems, i.e., smooth constant-rank control systems, is studied. The properties of the coefficients of a μ-commutative system and their interdependence are investigated.

A μ-commutative bilinear scalar control system is studied in detail. Using the dependence between the coefficients, a system is decomposed into "right" and "left" subsystems.

A direct representation for the input-output mapping of a μ-commutative bilinear scalar control system is derived and used to study the properties of the system, particularly the geometry of its reachable set. An example is given.

APPENDIX

Proof of Theorem 1. It is easily seen that the μ-commutativity conditions (5) ensure the bang-bang principle (4). Indeed, assuming in (5) that

$$\alpha = 0, \quad \beta = p,$$

we obtain

$$B_i \mathrm{ad}^p AB_j = \sum_{k=1}^m \mu_{ijk}^{0p} \mathrm{ad}^p AB_k, \tag{18}$$

and assuming in (5) that

$$\beta = 0, \quad \alpha = p,$$

we obtain

$$(\mathrm{ad}^p AB_i)B_j = \sum_{k=1}^{m} \mu_{ijk}^{p0} \mathrm{ad}^p AB_k.$$

Interchanging the subscripts i and j in the last equality and subtracting from (18) term by term, we obtain

$$[B_i, \mathrm{ad}^p AB_j] = \sum_{k=1}^{m} (\mu_{ijk}^{0p} - \mu_{jik}^{p0}) \mathrm{ad}^p AB_k,$$

i.e., conditions (4) are satisfied for

$$a_{\alpha\beta}^{ijk} = \begin{cases} \mu_{ij\beta}^{0k} - \mu_{ji\beta}^{k0}, & \alpha = k \\ 0, & \alpha \neq k. \end{cases}$$

Therefore, the bang-bang principle (4) holds for system (3), (5). Consequently, the system is controllable and of constant rank. This completes the proof of the theorem.

Proof of Theorem 2. Multiplying the equality

$$\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j = \sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mathrm{ad}^{\alpha+\beta} AB_k$$

on the right and left by the matrix A, let us subtract the relations thus obtained term by term:

$$A\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j - (\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j)A = \sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mathrm{ad}^{\alpha+\beta+1} AB_k,$$

but

$$\begin{aligned} &A\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j - (\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j)A \\ &= A\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j - (\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j)A + (\mathrm{ad}^\alpha AB_i)A\mathrm{ad}^\beta AB_j - (\mathrm{ad}^\alpha AB_i)A\mathrm{ad}^\beta AB_j \\ &= \Big(A\mathrm{ad}^\alpha AB_i - (\mathrm{ad}^\alpha AB_i)A\Big)\mathrm{ad}^\beta AB_j + \mathrm{ad}^\alpha AB_i\Big(A\mathrm{ad}^\beta AB_j - (\mathrm{ad}^\beta AB_j)A\Big) \\ &= [A, \mathrm{ad}^\alpha AB_i]\mathrm{ad}^\beta AB_j + \mathrm{ad}^\alpha AB_i[A, \mathrm{ad}^\beta AB_j] = \mathrm{ad}^{\alpha+1} AB_i \mathrm{ad}^\beta AB_j + \mathrm{ad}^\alpha AB_i \mathrm{ad}^{\beta+1} AB_j \\ &= \sum_{k=1}^{m} \mu_{ijk}^{\alpha+1,\beta} \mathrm{ad}^{\alpha+\beta+1} AB_k + \sum_{k=1}^{m} \mu_{ijk}^{\alpha,\beta+1} \mathrm{ad}^{\alpha+\beta+1} AB_k = \sum_{k=1}^{m} (\mu_{ijk}^{\alpha+1,\beta} + \mu_{ijk}^{\alpha,\beta+1}) \mathrm{ad}^{\alpha+\beta+1} AB_k. \end{aligned}$$

Hence

$$\sum_{k=1}^{m} \Big(\mu_{ijk}^{\alpha\beta} - (\mu_{ijk}^{\alpha+1,\beta} + \mu_{ijk}^{\alpha,\beta+1})\Big) \mathrm{ad}^{\alpha+\beta+1} AB_k = 0.$$

Since the matrices

$$\mathrm{ad}^{\alpha+\beta+1} AB_1, \mathrm{ad}^{\alpha+\beta+1} AB_2, \ldots, \mathrm{ad}^{\alpha+\beta+1} AB_m$$

are linearly independent, we obtain the unknown relation. This completes the proof of the theorem.

Proof of Theorem 3. We prove by induction on p. For $p = 0$, we have the relation

$$\mu_{ijk}^{00} = \mu_{ijk}^{00}.$$

Assuming that this relation holds for p, let us show that it holds for $p + 1$. By Theorem 2,

$$\mu_{ijk}^{00} = \sum_{q=0}^{p} C_p^q \mu_{ijk}^{q,p-q} = \sum_{q=0}^{p} C_p^q (\mu_{ijk}^{q+1,p-q} + \mu_{ijk}^{q,p-q+1})$$

$$= \sum_{q=0}^{p} C_p^q \mu_{ijk}^{q+1,p-q} + \sum_{q=0}^{p} C_p^q \mu_{ijk}^{q,p-q+1} = \sum_{q=1}^{p+1} C_p^{q-1} \mu_{ijk}^{q,p-q+1} + \sum_{q=0}^{p} C_p^q \mu_{ijk}^{q,p-q+1}$$

$$= \mu_{ijk}^{0,p+1} + \sum_{q=1}^{p} (C_p^{q-1} + C_p^q) \mu_{ijk}^{q,p-q+1} + \mu_{ijk}^{p+1,0}$$

$$= \mu_{ijk}^{0,p+1} + \sum_{q=1}^{p} C_{p+1}^q \mu_{ijk}^{q,p-q+1} + \mu_{ijk}^{p+1,0} = \sum_{q=0}^{p+1} C_{p+1}^q \mu_{ijk}^{q,p-q+1},$$

demonstrating the relation holds. This completes the proof of the theorem.

Proof of Theorem 4. Let us multiply the equality

$$\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j = \sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mathrm{ad}^{\alpha+\beta} AB_k$$

on the right by $\mathrm{ad}^\gamma AB_p$:

$$\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j \mathrm{ad}^\gamma AB_p = \sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mathrm{ad}^{\alpha+\beta} AB_k \mathrm{ad}^\gamma AB_p.$$

But

$$\mathrm{ad}^\alpha AB_i \mathrm{ad}^\beta AB_j \mathrm{ad}^\gamma AB_p = \mathrm{ad}^\alpha AB_i \sum_{k=1}^{m} \mu_{jpk}^{\beta\gamma} \mathrm{ad}^{\beta+\gamma} AB_k = \sum_{k=1}^{m} \sum_{q=1}^{m} \mu_{jpk}^{\beta\gamma} \mu_{ikq}^{\alpha,\beta+\gamma} \mathrm{ad}^{\alpha+\beta+\gamma} AB_q,$$

and

$$\sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mathrm{ad}^{\alpha+\beta} AB_k \mathrm{ad}^\gamma AB_p = \sum_{k=1}^{m} \sum_{q=1}^{m} \mu_{ijk}^{\alpha\beta} \mu_{kpq}^{\alpha+\beta,\gamma} \mathrm{ad}^{\alpha+\beta+\gamma} AB_q.$$

Hence

$$\sum_{q=1}^{m} \left(\sum_{k=1}^{m} \mu_{ijk}^{\alpha\beta} \mu_{kpq}^{\alpha+\beta,\gamma} - \sum_{k=1}^{m} \mu_{ikq}^{\alpha,\beta+\gamma} \mu_{jpk}^{\beta\gamma} \right) \mathrm{ad}^{\alpha+\beta+\gamma} AB_q = 0.$$

Since the matrices

$$\mathrm{ad}^{\alpha+\beta+\gamma} AB_1, \mathrm{ad}^{\alpha+\beta+\gamma} AB_2, \ldots, \mathrm{ad}^{\alpha+\beta+\gamma} AB_m$$

are linearly independent, we obtain the unknown relation. This completes the proof of the theorem.

REFERENCES

1. Agrachev, A.A. and Sachkov, Yu.L., *Geometricheskaya teoriya upravleniya* (Geometric Control Theory), Moscow: Fizmatlit, 2005.
2. Vakhrameev, S.A., Smooth Constant-Rank Control Systems and Linearizable Systems, in *Sovremennye problemy matematiki. Noveishie dostizheniya* (Modern Problems in Mathematics. Recent Advances), Moscow: VINITI, 1989, vol. 35, pp. 135–177.
3. Vakhrameev, S.A., Bang-Bang Theorems and Applied Problems, *Tr. Mat. Inst. Ross. Akad. Nauk*, 1998, vol. 220, pp. 49–112.
4. Topunov, M.V., The Reachable Set of a Quasi-Commutative Bilinear System: Its Convexity, *Avtom. Telemekh.*, 2003, no. 8, pp. 44–53.
5. Topunov, M.V., Convexity of the Reachable Set of a Quasicommutative Bilinear System, *Prikl. Mat. Mekh.*, 2003, vol. 63, no. 5, pp. 752–758.
6. Topunov, M.V., Convexity of Reachable Sets of a Smooth Linear Control System in Phase Variables, *Avtom. Telemekh.*, 2004, no. 11, pp. 79–85.
7. Topunov, M.V., Sufficient Conditions for Nesting the Reachable Sets of Two Smooth Control Systems of Constant Rank and Linear in Phase Variables, *Avtom. Telemekh.*, 2005, no. 12, pp. 114–125.
8. Davydov, A.A., Properties of the Reachable Bounds in Two-Dimensional Control Systems, *Uspekhi Mat. Nauk*, 1982, no. 37, no. 3, pp. 183–184.
9. Formal'skii, A.M., Corner Points of Bounds of Reachable Domains, *Prikl. Mat. Mekh.*, 1983, vol. 47, no. 4, pp. 566–574.
10. Chernous'ko, F.L., *Otsenivanie fazovogo sostoyaniya sistem. Metod ellipsoidov* (Phase State of Systems: An Ellipsoidal Estimation Method), Moscow: Nauka, 1988.
11. Sergeev, S.I., Reachability Sets of a Class of Multistep Control Processes: Their Design, *Avtom. Telemekh.*, 2002, no. 6, pp. 57–63.
12. Sirotin, A.N. and Formal'skii, A.M., Reachable Sets and Controllability of Linear Discrete Systems, *Ivz. Ross. Akad. Nauk, Teor. Syst. Upravlen.*, 2002, no. 4, pp. 5–16.
13. Vakhrameev, S.A., Geometrical and Topological Methods in Optimal Control Theory, *J. Math. Sci.*, 1995, vol. 76, no. 5, pp. 2555–2719.
14. Gantmakher, F.R., *Teoriya matrits*, Moscow: Nauka, 1988. Translated under the title *The Theory of Matrices*, New York: Chelsea, 1959.

This paper was recommended for publication by B.T. Polyak, a member of the Editorial Board