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DETERMINATE SYSTEMS

Ellipsoid-based Parametric Estimation in the Linear Multidimensional Systems with Uncertain Model Description[1]

S. A. Nazin and B. T. Polyak

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

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Abstract—Parametric estimation under uncertainty of the plant model description was considered within the framework of the ellipsoidal approach to the problems of guaranteed estimation. The unknown multidimensional plant whose parameter vector should be estimated was assumed to be linear and static, and uncertainty of its "input–output" model, to have both additive and multiplicative components. The external ellipsoidal approximations of the nonconvex information sets guaranteeing that the vector of possible plant parameters is contained in them were constructed from the results of observations. The method of their construction comes to semidefinite programming, that is, to minimization of the linear function constrained by the linear matrix inequalities which are readily realized by the numerical methods.

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1. INTRODUCTION

The alternative guaranteed approach which is based on the assumption of boundedness of the external perturbations and system errors and lack of information about their distribution laws [1–6] progresses hand in hand with the stochastic methods of estimation under uncertainty. It is namely this description of the model uncertainties that is most natural in many practical cases. In this context, the problems of estimation come to the determination of the set of all possible values of the desired variables such as the parameter vector of system phase coordinates compatible with the given constraints on uncertainty.

Along with the advantages such as the need to know only the upper and lower boundaries of the uncertain variables, the guaranteed approach has essential disadvantages such as computational laboriousness of its methods or conservatism of the resulting estimates which hinder its use in applications. For example, the interval formulations of the problems of estimation often prove to be NP-hard; their simplification appreciably impairs the accuracy of the resulting estimates. In this connection, the method of ellipsoids is one of the most convenient and attractive approaches to guaranteed estimation. Within its framework, the system perturbations and errors are assumed to meet the quadratic ellipsoidal constraints, and a problem is posed of determining the ellipsoid as an estimate of the unknown parameter vector or the vector of system phase coordinates. At that we note that each point of this ellipsoid may be equally regarded as the solution of the original problem of estimation. For the time being, the method of ellipsoids is well suited for the case of the linear dynamic systems [4, 7–9] and, in particular, for analysis and design of control.

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The present paper is devoted to the parametric estimation in the multidimensional systems under uncertainties which lies in determining the boundaries or multiple constraints on the parameter vector from the results of observations which is one of the most natural problems in this context. Let us consider the linear "input–output" model of the static multidimensional plant

$$y = C\,x + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of unknown plant parameters, $y \in \mathbb{R}^m$ is the vector of measurements (outputs), $C \in \mathbb{R}^{m\times n}$ is the matrix of model regressors (inputs), and $w \in \mathbb{R}^m$ is the vector of measurement errors. We assume that in the matrix C there is no autoregression component, that is, that C is independent of y. It is desired to determine the guaranteed estimate of the vector x on the basis of the given y and C and the constraints on w in (1).

The classical approach to the plant parameter estimation relies on the assumption that the matrix of the regressors C is defined precisely in the model and the vector w is bounded, for example, in the Euclidean norm $\|w\| \leqslant \delta$ or componentwise $|w_i| \leqslant \delta_i$, $i = 1, \ldots, m$. The papers [10, 11] proposing recurrent ellipsoidal algorithms of parametric identification of the linear systems are accepted as the fundamental publications in this area. It is common knowledge that within the framework of the ellipsoidal technique the problem of parameter estimation comes to approximation of the intersection of ellipsoids which in its optimal formulation is NP-hard. Yet at the same time the search of suboptimal approximations of the ellipsoid intersection can be performed rather easily [9, 12] without appreciable losses in the quality of estimates. In this context, it may be useful to mention the special issue of *Mathematics and Computers in Simulation* devoted to parameter identification with error bound [13], two reviews [14, 15] on this subject, paper [16] on identification of the nonlinear systems, as well as the collected papers on the guaranteed identification [17] which covers a wide range of questions and methods on parametric estimation under unknown bounded noise.

The assumption of precise knowledge of the model regressors, however, is often impracticable because of lack of the precise information about the structure of a physical plant. Therefore, an important role is played here by the robust principles and approaches taking into account the unavoidable uncertainty of the model description, which gives rise to the matrix multiplicative uncertainty in (1). For the same formulation, some cases of using the interval technique for parametric estimation were considered in [13, 18, 19] where the difficulties concerned with NP-hardness of the problem were underlined. The latest findings on the interval methods of estimation and their various applications can be found in the new monograph [20]. As an alternative to the interval method, an ellipsoidal approach to estimation of the phase states of the dynamic systems under model uncertainty which reduces the problem to one-dimensional minimization was proposed in [21, 22]. On the other hand, the method of ellipsoids is especially handy from the standpoint of the technique of linear matrix inequalities [23] and subsequent consideration of the more general structured matrix uncertainty in the model [24], which allows one to reduce the search of the optimal ellipsoidal estimates to the so-called problem of semidefinite programming [23, 25], that is, to the minimization of the linear function constrained by linear matrix inequalities that are readily realized by the computational approaches.

The aim of the present paper is to generalize the results of [22] and use the technique of the linear matrix inequalities in order to propose a new method of ellipsoidal estimation of the parameter vector of the static multidimensional plant under uncertainty in the description of its model. The basic novelty of this method lies in that the estimates of the parameters of the systems with multiple measurements (outputs) are constructed nonrecurrently using the procedure of semidefinite programming, which often enables one to avoid the negative effect of error accumulation and enhance appreciably the quality of the estimates obtained.

The problem and some auxiliary statements are formulated in Sections 2 and 3, respectively. The single-measurement (scalar-output) model with uncertainty is presented in Section 4 for illustration. For the multidimensional systems, Section 5 proposes an algorithm of ellipsoidal parametric estimation. Section 6 discusses the recurrent and direct estimation algorithms presented, respectively, in Sections 4 and 5. The paper is concluded by a brief summary.

2. FORMULATION OF THE PROBLEM

Let some a priori multiple estimate of the plant parameter vector, which often can be too large and unsatisfactory, be known from the beginning. We also assume that at some time instant an arbitrary collection of the experimental "input–output" data (observations) of the plant is accessible. It is desired to enhance the a priori estimate on the basis of observations.

So, let $x \in \mathcal{E}_0 = \mathcal{E}(c_0, P_0)$ with the matrix $P_0 > 0$, where $\mathcal{E}_0$ is an a priori bounded ellipsoidal estimate of the unknown parameter vector x. The ellipsoid in $\mathbb{R}^n$ is denoted by $\mathcal{E}(c,P) = \left\{x \in \mathbb{R}^n :\ (x-c)^{\mathrm{T}}P(x-c) \leqslant 1\right\}$, where c is the center of the ellipsoid and $P = P^{\mathrm{T}} > 0$ is its matrix defining the form and size of the ellipsoid. We note that the case of the matrix $P \geqslant 0$ corresponds to the degenerate unbounded ellipsoid.

Let us consider the linear model of a plant with uncertainty in data

$$y_k = (a_k + \Delta a_k)^{\mathrm{T}}\, x + w_k, \quad k = 1, \dots, m, \tag{2}$$

where $x \in \mathbb{R}^n$, $y = (y_1, \dots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$; a_k is the column vector consisting of the kth row of the regressor matrix C; the variable w_k describes the error in the value of the output y_k and is the additive uncertainty in (2). The vector Δa_k represents the uncertainty in the description of the plant model and is the multiplicative component of uncertainty in (2). It may be also considered directly as an error in the value of the regressor (input vector) a_k. We assume in what follows that all errors in the model are unknown but bounded. Within the framework of the ellipsoidal technique, it is convenient to assume that the multiplicative and additive components of uncertainty satisfy the common quadratic constraints

$$\frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1, \quad k = 1, \dots, m. \tag{3}$$

At that, the case of $\varepsilon_k = 0$ implies that $\Delta a_k = 0$ and $|w_k| \leqslant \delta_k$, and the case of $\delta_k = 0$ implies that, respectively, $w_k = 0$ and $\|\Delta a_k\| \leqslant \varepsilon_k$. Here and below, $\|\cdot\|$ is the Euclidean norm of the vector. The nominal values of y_k and a_k and the numbers ε_k and δ_k are assumed to be known. Then, all vectors x satisfying (2) under the constraints (3) make up the so-called *information set* $\mathcal{X}$ of the system, that is, the set of every possible values of the plant parameters compatible with the results of observations

$$\mathcal{X} = \left\{x \in \mathbb{R}^n \colon y_k = (a_k + \Delta a_k)^{\mathrm{T}}\, x + w_k, \frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1, k = 1, \dots, m\right\}. \tag{4}$$

If in the model all errors meet the aforementioned constraints, then the real value of the parameter vector x^* belongs to $\mathcal{X}$. In the general case, its precise description is hindered by complexity of the information set, nonconvexity for $\Delta a_k \neq 0$, and possible nonsimple connectedness. Therefore, we seek its external approximations by simpler domains such as ellipsoids. Since from the very beginning it was assumed that some a priori estimate $\mathcal{E}_0 = \mathcal{E}(c_0, P_0)$ is given, it is advisable to consider $x \in \mathcal{E}_0 \cap \mathcal{X}$ as the estimate of x^*. The problem lies in determining as small as possible ellipsoid $\mathcal{E} = \mathcal{E}(d, Q)$ comprising the intersection

$$\mathcal{E} \supseteq \mathcal{E}_0 \cap \mathcal{X}. \tag{5}$$

All its points will be the desired estimate of the plant parameter vector.

We note that it is assumed in what follows that $\mathcal{X} \neq \emptyset$ and $x^* \in \mathcal{X}$, that is, the physical plant under consideration is adequately described by the model (2), (3) and in the experimental data there are no so-called outliers (false measurements whose errors exceed the predetermined limits).

3. AUXILIARY STATEMENTS

In this section we formulate some auxiliary statements that will be used further.

Lemma 1. *Let $\zeta = u^{\mathrm{T}}x + v$, where $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}$. Then, in order that*

$$\frac{\|u\|^2}{\varepsilon^2} + \frac{|v|^2}{\delta^2} \leqslant 1$$

for any fixed x, it is necessary and sufficient to satisfy the inequality $|\zeta|^2 \leqslant \varepsilon^2\|x\|^2 + \delta^2$.

Proof can be found in [22], but it is cited in the Appendix for completeness sake.

Therefore, the joint quadratic constraint on u and v provides a quadratic constraint for ζ. On the contrary, consideration of individual constraints like $\|u\| \leqslant \varepsilon$ and $|v| \leqslant \delta$ leads to an equivalent, but already nonquadratic inequality $|\zeta| \leqslant \varepsilon\|x\| + \delta$.

Now we consider an arbitrary finite collection of, possibly, degenerate ellipsoids $\mathcal{E}_k = \mathcal{E}(c_k, P_k)$, $k = 1, \ldots, K$. If their intersection $\mathcal{I}_K = \bigcap_{k=1}^{K} \mathcal{E}(c_k, P_k)$ is not empty, then it is convex. Since search of the optimal ellipsoid comprising this intersection is NP-hard, in many cases one contents oneself with simpler suboptimal estimates. For external approximation, optimality of the ellipsoidal estimates is understood from the standpoint of the minimal size of ellipsoid. The minimality criteria for volume and trace corresponding to the weight functions

$$f_1(P) = -\ln \det P, \quad f_2(P) = \operatorname{tr} P^{-1},$$

where P is the ellipsoid matrix, pass for the most popular ones. More general criteria are discussed in [7]. The following lemma defines the family of ellipsoidal approximations comprising the nonempty intersection of two or more ellipsoids.

Lemma 2. *Let $\mathcal{I}_K = \bigcap_{k=1}^{K} \mathcal{E}(c_k, P_k)$, where $P_k \geqslant 0$, $k = 1, \ldots, K$, and $\mathcal{I}_K \neq \emptyset$, and let $\mathcal{A}_K$ be the set of vectors $\alpha = (\alpha_1, \ldots, \alpha_K)^{\mathrm{T}} \in \mathbb{R}^K$ such that $\alpha_1 \geqslant 0, \ldots, \alpha_K \geqslant 0$ and $\sum_{k=1}^{K} \alpha_k = 1$. We consider*

$$P_\alpha = \sum_{k=1}^{K} \alpha_k P_k, \quad c_\alpha = P_\alpha^{-1} \sum_{k=1}^{K} \alpha_k P_k c_k, \quad \xi_\alpha = \sum_{k=1}^{K} \alpha_k c_k^{\mathrm{T}} P_k c_k - c_\alpha^{\mathrm{T}} P_\alpha c_\alpha.$$

Then,

$$\mathcal{E}\left(c_\alpha, (1 - \xi_\alpha)^{-1} P_\alpha\right) \supseteq \mathcal{I}_K$$

for all $\alpha = (\alpha_1, \ldots, \alpha_K)^{\mathrm{T}} \in \mathcal{A}_K$ such that $P_\alpha > 0$.

Proof is given in [9]. Therefore, we skip it, but it can be readily carried out along the lines of the proof of Theorem 1.

We note that the requirement of $P_\alpha > 0$ corresponds to the condition for nondegeneracy (boundedness) of the resulting ellipsoid because the multiplier $(1 - \xi_\alpha)^{-1}$ proves to be positive for any $\alpha \in \mathcal{A}_K$. Therefore, the search of the suboptimal ellipsoidal estimate comprising the intersection $\mathcal{I}_K$, can be reduced to the search of the minimal ellipsoid among the family of Lemma 2. We note, however, that the authors of [9] failed to establish convexity of the corresponding weight functions $\varphi_1(\alpha) = -\ln\det\dfrac{P_\alpha}{1-\xi_\alpha}$ and $\varphi_2(\alpha) = (1-\xi_\alpha)\,\mathrm{tr}\,P_\alpha^{-1}$. Nevertheless, by considering simpler dependences of α, one can reduce the search of the minimal ellipsoid in the family of Lemma 2 to the problem of convex minimization.

Lemma 3. *Under conditions of Lemma 2, it is also true that*

$$\mathcal{E}\,(c_\alpha, P_\alpha) \supseteq \mathcal{I}_K$$

for all $\alpha = (\alpha_1, \ldots, \alpha_K)^{\mathrm{T}} \in \mathcal{A}_K$ *such that* $P_\alpha > 0$. *At that, the functions* $\widehat{\varphi}_1(\alpha) = -\ln\det P_\alpha$ *and* $\widehat{\varphi}_2(\alpha) = \mathrm{tr}\,P_\alpha^{-1}$ *are convex on* $\mathcal{A}_K$.

Therefore, Lemmas 2 and 3 offer two families of the ellipsoidal approximations $\mathcal{E}(c_\alpha, (1-\xi_\alpha)^{-1}P_\alpha)$ and $\mathcal{E}\,(c_\alpha, P_\alpha)$ comprising the intersection $\mathcal{I}_K$, the second family being convex in α on $\mathcal{A}_K$. However, the minimal ellipsoid of the first family provides a better approximation of the intersection. It was noted in [9] that the difference between the minimal ellipsoids of these two families is mostly insignificant. In the present paper, Lemma 2 will be extended to the case of intersection of the quadratic nonellipsoidal sets arising at estimation of the linear system parameters.

4. SINGLE-OUTPUT SYSTEM

From the point of view of analysis of the information sets, it is advisable to consider the case of one measurement in the system, that is, of the multiple input–single output model ($m = 1$)

$$y_1 = (a_1 + \Delta a_1)^{\mathrm{T}}\, x + w_1, \tag{6}$$

where $x \in \mathbb{R}^n$, $y_1 \in \mathbb{R}$, and

$$\frac{\|\Delta a_1\|^2}{\varepsilon_1^2} + \frac{|w_1|^2}{\delta_1^2} \leqslant 1. \tag{7}$$

According to Lemma 1,

$$|y_1 - a_1^{\mathrm{T}}x|^2 \leqslant \varepsilon_1^2\|x\|^2 + \delta_1^2 \tag{8}$$

or, which is equivalent,

$$x^{\mathrm{T}}\left(a_1a_1^{\mathrm{T}} - \varepsilon_1^2 I\right)x - 2y_1a_1^{\mathrm{T}}x + y_1^2 - \delta_1^2 \leqslant 0. \tag{9}$$

We denote by I the identity matrix and assume that $0 < \varepsilon_1 < \|a_1\|$, which is always the case if the multiplicative error $\Delta a_1 \neq 0$ in the model is known to be smaller than the corresponding nominal value of the regressor a_1. Then, the matrix $a_1a_1^{\mathrm{T}} - \varepsilon_1^2 I$ is invertible and inequality (9) is representable as the following quadratic form:

$$(x - g_1)^{\mathrm{T}} M_1 (x - g_1) \leqslant 1, \tag{10}$$

where

$$\left[\begin{array}{l} M_1 = R_1/\left(y_1^2 a_1^{\mathrm{T}} R_1^{-1} a_1 - y_1^2 + \delta_1^2\right) \\ g_1 = y_1 R_1^{-1} a_1 \\ R_1 = a_1 a_1^{\mathrm{T}} - \varepsilon_1^2 I. \end{array}\right. \tag{11}$$

For $\varepsilon_1 > 0$, the matrices R_1 and M_1 turn out to be of uncertain sign. Therefore, the information set

$$\mathcal{X}_1 = \left\{x \in \mathbb{R}^n:\ \ (x - g_1)^{\mathrm{T}} M_1 (x - g_1) \leqslant 1\right\} \tag{12}$$

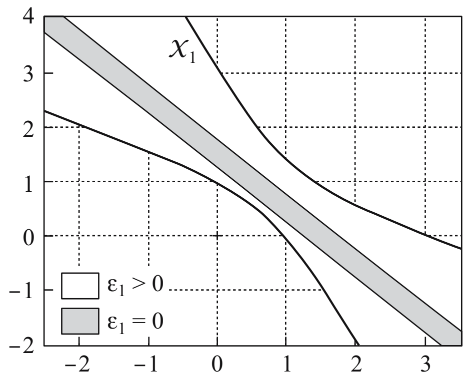

Fig. 1. Information set $\mathcal{X}_1$.

made up by all vectors x satisfying (10) is not an ellipsoid in $\mathbb{R}^n$, but rather is unbounded and nonconvex. Figure 1 depicts an example of such a set for the two-dimensional case ($n = 2$), where $\mathcal{X}_1$ is the space between two hyperbolas on a plane. At the same time, for $\varepsilon_1 = 0$ the domain $\mathcal{X}_1$ would be a strip (shaded area in Fig. 1), which in turn is the classical information set for the scalar measurement in the absence of the matrix uncertainty.

With a priori knowledge of the ellipsoidal estimate $\mathcal{E}_0$ for x, it is possible to construct an ellipsoid approximating the intersection $\mathcal{E}_0 \cap \mathcal{X}_1$.

Lemma 4. *If the vector $x \in \mathcal{E}(c_0, P_0)$, $P_0 > 0$ and satisfies $y_1 = (a_1 + \Delta a_1)^{\mathrm{T}} x + w_1$ for the constraint* (7) *on Δa_1 and w_1, then $x \in \mathcal{E}(d_\tau, (1 - \xi_\tau)^{-1} Q_\tau)$, where*

$$\left[\begin{array}{l} Q_\tau = (1-\tau)P_0 + \tau M_1 \\ d_\tau = Q_\tau^{-1}[(1-\tau)P_0 c_0 + \tau M_1 g_1] \\ \xi_\tau = (1-\tau)c_0^{\mathrm{T}} P_0 c_0 + \tau g_1^{\mathrm{T}} M_1 g_1 - d_\tau^{\mathrm{T}} Q_\tau d_\tau \end{array}\right. \tag{13}$$

for all values of τ from the interval $0 < \tau < \tau_{\max} = \min\{1, (1-\lambda_{\min})^{-1}\}$, where $\lambda_{\min}$ is the minimal generalized eigenvalue of the matrix pair (M_1, P_0). The values of M_1 and g_1 are defined in (11).

The eigenvalue λ_i and the eigenvector v_i of the matrix pair (M, P) are referred to as generalized if $M v_i = \lambda_i P v_i$.

Proof of Lemma 4 is given in [22]. Therefore, we skip it, but it can be readily carried out along the lines of the proof of Theorem 1.

Lemma 4 gives a single-parameter family of the ellipsoids $\mathcal{E}(d(\tau), Q(\tau))$ comprising the intersection $\mathcal{E}_0 \cap \mathcal{X}_1$. Therefore, from the standpoint of volume $\varphi_1(\tau) = -\ln\det \dfrac{Q_\tau}{1-\xi_\tau}$ or trace $\varphi_2(\tau) = (1-\xi_\tau)\operatorname{tr} Q_\tau^{-1}$, search of the minimal ellipsoid in this family comes to the problem of one-dimensional minimization and provides the suboptimal estimate for $\mathcal{E}_0 \cap \mathcal{X}_1$ which, as can be shown on the basis of the results of [26], proves to be optimal only in the special case where $c_0 = g_1$. We note that the functions $\varphi_1(\tau)$ and $\varphi_2(\tau)$, seem to be strictly convex in τ over the interval $0 < \tau < \tau_{\max}$ as it is the case with approximation of ellipsoid intersection (Lemma 2). This fact is born out by numerous experiments, but its strict proof still was not established.

On the other hand, by considering simpler convex objective functions such as $\widehat{\varphi}_1(\tau) = -\ln\det Q_\tau$ or $\widehat{\varphi}_2(\tau) = \operatorname{tr} Q_\tau^{-1}$, the problem of determining the suboptimal estimate of the intersection $\mathcal{E}_0 \cap \mathcal{X}_1$ among the family of Lemma 4 can be reduced to the convex problem. However, it is obvious that in this case the determined ellipsoid will not be minimal either in volume or trace in the given

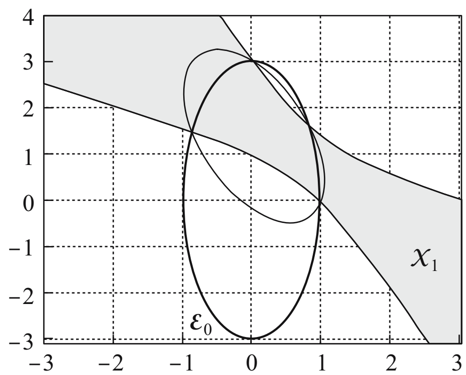

Fig. 2. Approximation of the intersection $\mathcal{E}_0 \cap \mathcal{X}_1$.

family. In the majority of cases, nevertheless, its difference with the minimal ellipsoid in terms of one or another criterion is negligible, as is illustrated by the following example.

Example 1. Let $m = 1$, $n = 2$. We take $y_1 = 3$, $a_1 = (2,\ 2)^{\mathrm{T}}$, $\varepsilon_1 = 1$ and $\delta_1 = 0.5$. In this case, two eigenvalues of the matrix $R_1 = a_1 a_1^{\mathrm{T}} - \varepsilon_1^2 I$ from (11) have opposite signs. Thus, neither R_1 nor M_1 is positive or negative definite. Therefore, the information set $\mathcal{X}_1$ of (12) is the space between two hyperbolas on the plane (see Fig. 2). We assume that it is known in advance that the system phase vector x belongs to some nondegenerate ellipsoid $\mathcal{E}(c_0, P_0)$ centered at zero $c_0 = 0$ and having the matrix $P_0 = \operatorname{diag}\{1,\ 1/9\}$. By calculating $\lambda_{\min} = -1.2832$ and $\tau_{\max} = 0.4380 < 1$ from Lemma 4, we determine the ellipsoid $\mathcal{E}(d, Q)$ which is minimal in the criterion of volume or trace in the single-parameter family (13). This may be done easily because in this case the functions $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are convex. The solid line in Fig. 2 shows the intersection approximation in the trace criterion by the minimal ellipsoid. For comparison, the dashed line in the same figure shows the ellipsoid minimizing the simple convex function $\widehat{\varphi}_2(\tau) = \operatorname{tr} Q_\tau^{-1}$. We note the negligible difference between these two ellipsoidal approximations. $\square$

In the case of multi-output models, the problem of approximation of the occurring information set may be solved recurrently by considering successively the scalar cases and applying Lemma 4. But this procedure introduces additional errors at each recursion step and becomes inconvenient for multiple-output models. For the multidimensional case, the next section proposes a nonrecurrent approach to construction of the ellipsoidal estimates which is based on the linear matrix inequalities.

5. MULTIDIMENSIONAL SYSTEMS

Now we proceed to the linear model (2) of the multiple-input multiple-output plant

$$y_k = (a_k + \Delta a_k)^{\mathrm{T}} x + w_k, \qquad k = 1, \ldots, m,$$

where the multiplicative and additive components of uncertainty satisfy the joint constraints (3)

$$\frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1, \qquad k = 1, \ldots, m.$$

Then, the information set (4) will be formed by the intersection $\mathcal{X} = \bigcap\limits_{k=1}^{m} \mathcal{X}_k$, where

$$\mathcal{X}_k = \left\{ x \in \mathbb{R}^n : \ \ y_k = (a_k + \Delta a_k)^{\mathrm{T}} x + w_k, \ \ \frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1 \right\} \tag{14}$$

is the information set of the system with the scalar output y_k. By virtue of nonconvexity of $\mathcal{X}_k$, the set $\mathcal{X}$ will be nonconvex and sometimes nonsimply connected. Similar to the reasoning of the last section, we represent

$$\mathcal{X}_k = \left\{x \in \mathbb{R}^n : \quad (x - g_k)^{\mathrm{T}} M_k (x - g_k) \leqslant 1\right\}, \tag{15}$$

where

$$\left[\begin{array}{l} M_k = R_k / \left(y_k^2 a_k^{\mathrm{T}} R_k^{-1} a_k - y_k^2 + \delta_k^2\right) \\ g_k = y_k R_k^{-1} a_k \\ R_k = a_k a_k^{\mathrm{T}} - \varepsilon_k^2 I. \end{array}\right. \tag{16}$$

Let there be an a priori estimate $\mathcal{E}_0 = \mathcal{E}(c_0, P_0)$ of the parameter vector x. We assume that $P_0 > 0$ and formulate the main result.

Theorem 1. *If the vector* $x \in \mathcal{E}(c_0, P_0)$, $P_0 > 0$ *and satisfies*

$$y_k = (a_k + \Delta a_k)^{\mathrm{T}} x + w_k, \qquad \frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1, \quad k = 1, \ldots, m,$$

then $x \in \mathcal{E}(d_\alpha, (1 - \xi_\alpha)^{-1} Q_\alpha)$, *where*

$$\left[\begin{array}{l} Q_\alpha = \alpha_0 P_0 + \sum\limits_{k=1}^{m} \alpha_k M_k \\ d_\alpha = Q_\alpha^{-1} \left(\alpha_0 P_0 c_0 + \sum\limits_{k=1}^{m} \alpha_k M_k g_k\right) \\ \xi_\alpha = \alpha_0 c_0^{\mathrm{T}} P_0 c_0 + \sum\limits_{k=1}^{m} \alpha_k g_k^{\mathrm{T}} M_k g_k - d_\alpha^{\mathrm{T}} Q_\alpha d_\alpha, \end{array}\right. \tag{17}$$

for all values of $\alpha = (\alpha_0, \ldots, \alpha_m)^{\mathrm{T}}$ *such that* $\alpha_k \geqslant 0$, $k = 0, \ldots, m$, $\sum\limits_{k=0}^{m} \alpha_k = 1$ *and* $Q_\alpha > 0$.

Proof of the theorem can be found in the Appendix.

Theorem 1 gives a family of ellipsoidal approximations comprising the intersection $\mathcal{E}_0 \cap \mathcal{X}$. This family generalizes Lemma 2 to the case of intersection of nonellipsoidal sets and Lemma 4 to the multidimensional systems. Minimization of the functions $\varphi_1(\alpha) = -\ln\det \dfrac{Q_\alpha}{1 - \xi_\alpha}$ or $\varphi_2(\alpha) = (1 - \xi_\alpha) \operatorname{tr} Q(\alpha)^{-1}$ on this family provides the ellipsoidal estimates of the parameter vector which are suboptimal in volume and trace, respectively. However, it seems that nothing can be said about convexity of these functions. Nevertheless, consideration may be given to simpler objective functions such as $\widehat{\varphi}_1(\tau) = -\ln\det Q_\tau$ or $\widehat{\varphi}_2(\tau) = \operatorname{tr} Q_\tau^{-1}$; then, the problem of determining the suboptimal estimate of the intersection $\mathcal{E}_0 \cap \mathcal{X}$ in the given family will be convex.

We denote by $\widehat{\mathcal{A}}_m$ the set of all $\alpha = (\alpha_1, \ldots, \alpha_m)^{\mathrm{T}}$ satisfying the conditions of Theorem 1:

$$\widehat{\mathcal{A}}_m = \left\{\alpha = (\alpha_0, \ldots, \alpha_m)^{\mathrm{T}} : \quad \alpha_k \geqslant 0, \ k = 0, \ldots, m, \ \sum_{k=0}^{m} \alpha_k = 1, \ \ Q_\alpha > 0\right\}.$$

The problem of minimizing the function $\widehat{\varphi}_1(\alpha)$ or $\widehat{\varphi}_2(\alpha)$ on $\widehat{\mathcal{A}}_m$ is that of convex optimization. Let us consider in more detail the minimization by the trace criterion. Since trace is a linear function, the problem can be reduced to a linear one. Indeed, to determine the trace-minimal ellipsoid in the family of Theorem 1, one has to solve

$$\min \operatorname{tr} Q_\alpha^{-1} \tag{18}$$

under the constraints

$$\left[\begin{array}{l} \alpha_k \geqslant 0, \quad k = 0, \ldots, m \\ \sum\limits_{k=0}^{m} \alpha_k = 1 \\ Q_\alpha = \alpha_0 P_0 + \sum\limits_{k=1}^{m} \alpha_k M_k > 0. \end{array}\right. \tag{19}$$

Conditions (19) represent linear constraints of which the last one is a linear matrix inequality. Additionally, minimization of the convex nonlinear function (18) can be reduced to a linear problem. Indeed, let e_i be the ith unit vector in the space $\mathbb{R}^n$. Then, $\operatorname{tr} Q_\alpha^{-1} = \sum\limits_{i=1}^{n} e_i^{\mathrm{T}} Q_\alpha^{-1} e_i$. Since according to the Schur formula for the block matrices (see [27])

$$\mu_i \geqslant e_i^{\mathrm{T}} Q_\alpha^{-1} e_i \quad \Longleftrightarrow \quad \begin{pmatrix} \mu_i & e_i^{\mathrm{T}} \\ e_i & Q_\alpha \end{pmatrix} \geqslant 0,$$

$\operatorname{tr} Q_\alpha^{-1} \leqslant \sum\limits_{i=1}^{n} \mu_i$. Consequently, problem (18) under conditions (19) is equivalent to the linear problem

$$\min \sum_{i=1}^{n} \mu_i \tag{20}$$

under the constraints

$$\left[\begin{array}{l} \alpha_k \geqslant 0, \quad k = 0, \ldots, m \\ \sum\limits_{k=0}^{m} \alpha_k = 1 \\ Q_\alpha = \alpha_0 P_0 + \sum\limits_{k=1}^{m} \alpha_k M_k > 0 \\ \begin{pmatrix} \mu_i & e_i^{\mathrm{T}} \\ e_i & Q_\alpha \end{pmatrix} \geqslant 0, \quad i = 1, \ldots, n. \end{array}\right. \tag{21}$$

Therefore, the search of the suboptimal ellipsoidal estimate by the trace criterion came to the linear minimization problem under constraints like linear matrix inequalities. These problems are called the problems of semidefinite programming for which a number of solvers exists such as `LMI Toolbox` in the `MATLAB` system which solve them effectively.

To illustrate the aforementioned procedure, we consider the case of a two-dimensional vector of system parameters and three measurements such that the information set $\mathcal{X}$ made up by them is nonempty and bounded. The set $\mathcal{X}$ is the intersection of three information sets of the system with scalar outputs, that is, $\mathcal{X} = \bigcap\limits_{i=1}^{3} \mathcal{X}_i$. Such situation is exemplified in Fig. 3 where $\mathcal{X}$ (shaded area) is

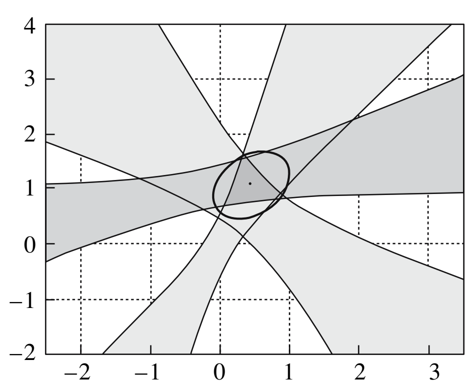

Fig. 3. Approximation of intersection.

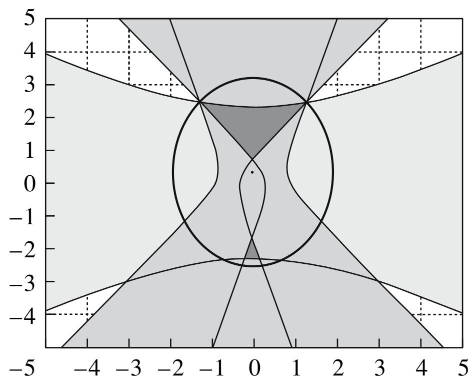

Fig. 4. Approximation of intersection.

made up by the intersection of $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$. Since the intersection $\mathcal{X}$ is bounded, the condition $Q_\alpha = \sum_{k=1}^{3} \alpha_k M_k > 0$ will be satisfied under some $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$ such that $\sum_{k=0}^{m} \alpha_k = 1$. Therefore, in this case one can do without considering the a priori ellipsoidal estimate $\mathcal{E}(c_0, P_0)$ with $P_0 > 0$. By solving the problem of minimization (20) with the constraints (21) where $\alpha_0 \equiv 0$, we obtain a suboptimal ellipsoidal approximation for $\mathcal{X}$ shown by the bold line in Fig. 3. The bounded information set $\mathcal{X}$ often may be nonsimply connected as shown, for example, in Fig. 4. Nevertheless, solution of problem (20), (21) provides an ellipsoid comprising this set. The next section compares the direct and recurrent methods of ellipsoidal parametric estimation.

6. RECURRENT AND DIRECT APPROACHES: DISCUSSION

In the multidimensional models, the problem of approximation of the arising information sets can be solved recurrently by considering successively the scalar cases and applying the technique of Section 4. Although this procedure is undoubtedly simple because each individual measurement requires solution of a one-dimensional convex problem of minimization, at each step of recursion it introduces additional errors, which for the multiple-input multiple-output systems makes this method unattractive, and the resulting estimates become very conservative. Therefore, of great importance is the direct approach which is reducible to the linear matrix inequalities and the problem of semidefinite programming that can be readily solved actually by means of any packet of convex optimization. To compare the direct and recurrent methods of ellipsoidal estimation, we consider the following examples.

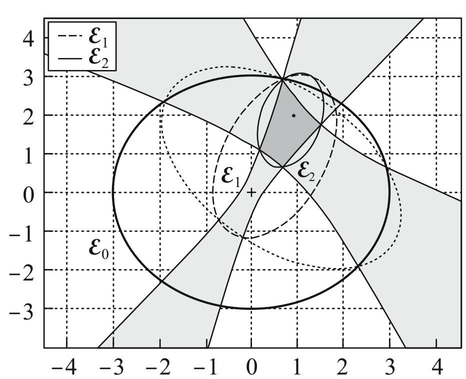

Fig. 5. Approximation of intersection by the recurrent and direct methods.

Example 2. Let $\mathcal{E}_0 = \mathcal{E}(0, I/9)$, where I is the identity matrix, be a sphere of radius $r = 3$ which is known to comprise the two-dimensional plant parameter vector x. Let also two measurements in model (2) with the data $y_1 = 2$, $a_1 = (1,1)^{\mathrm{T}}$, $\varepsilon_1 = \delta_1 = 0.5$ and $y_2 = 0$, $a_2 = (2,-1)^{\mathrm{T}}$, $\varepsilon_2 = \delta_2 = 0.5$ be given. The information set $\mathcal{X}$ is shown for this case in Fig. 5 (shaded area). Recurrent application of Lemma 4 provides an ellipsoidal estimate $\mathcal{E}_1$ (dashed line) which appreciably exceeds the approximation $\mathcal{E}_2$ (solid line in Fig. 5) obtained by solving the problem of semidefinite programming (20), (21). We note that the difference between the estimates grows with the number of measurements. □

7. CONCLUSIONS

A method of ellipsoidal estimation of the parameters of the linear stationary systems with uncertain description of the model was proposed. This approach relies on the technique of linear matrix inequalities which enables one to estimate without unnecessary conservatism the parameters of the multidimensional systems by means of the numerical solution of the problem of semidefinite programming. The approach proposed can find application, for example, in robot engineering and navigation, processing of signals and images, as well as in other applications where the uncertain variables in the models are assumed to be unknown, but bounded.

APPENDIX

Proof of Lemma 1. If

$$\frac{\|u\|^2}{\varepsilon^2} + \frac{|v|^2}{\delta^2} \leqslant 1,$$

then $|\zeta| = |u^{\mathrm{T}}x + v| \le \|u\|\|x\| + |v| \leqslant (\varepsilon^2\|x\|^2 + \delta^2)^{1/2}$.

On the other hand, if $|\zeta|^2 \leqslant \varepsilon^2\|x\|^2 + \delta^2$, then by taking

$$u = \frac{\varepsilon^2}{\varepsilon^2\|x\|^2 + \delta^2}\,\zeta\, x, \quad v = \frac{\delta^2}{\varepsilon^2\|x\|^2 + \delta^2}\,\zeta$$

we get that $u^{\mathrm{T}}x + v = \zeta$ and $\dfrac{\|u\|^2}{\varepsilon^2} + \dfrac{|v|^2}{\delta^2} \leqslant 1$, which proves the lemma. □

Proof of Theorem 1. Let

$$x \in \mathcal{E}(c_0, P_0), \quad P_0 > 0, \tag{A.1}$$

and

$$y_k = (a_k + \Delta a_k)^{\mathrm{T}}x + w_k, \quad \frac{\|\Delta a_k\|^2}{\varepsilon_k^2} + \frac{|w_k|^2}{\delta_k^2} \leqslant 1, \quad k = 1, \dots, m \tag{A.2}$$

be true. Then, the inequalities

$$(x - c_0)^{\mathrm{T}}P_0(x - c_0) \leqslant 1, \quad P_0 > 0,$$
$$(x - g_k)^{\mathrm{T}}M_k(x - g_k) \leqslant 1, \quad k = 1, \dots, m,$$

where M_k and g_k are from (16) and the matrix M_k is not sign-definite, are satisfied equivalently. Let us consider $\alpha = (\alpha_0, \dots, \alpha_m)^{\mathrm{T}}$ such that $\alpha_k \geqslant 0$, $k = 0, \dots, m$, and $\sum\limits_{k=0}^{m} \alpha_k = 1$. We take a linear combination of the quadratic forms

$$\alpha_0(x - c_0)^{\mathrm{T}}P_0(x - c_0) + \sum_{k=1}^{m}\Big[\alpha_k(x - g_k)^{\mathrm{T}}M_k(x - g_k)\Big] \leqslant \alpha_0 + \sum_{k=1}^{m}\alpha_k \leqslant 1$$

and obtain after simple transformations that

$$\alpha_0(x-c_0)^{\mathrm{T}}P_0(x-c_0)+\sum_{k=1}^{m}\Big[\alpha_k(x-g_k)^{\mathrm{T}}M_k(x-g_k)\Big]=(x-d_\alpha)^{\mathrm{T}}Q_\alpha(x-d_\alpha)+\xi_\alpha,$$

where d_α, Q_α, and ξ_α are from (17), whence the inequality

$$(x-d_\alpha)^{\mathrm{T}}Q_\alpha(x-d_\alpha)\leqslant 1-\xi_\alpha \tag{A.3}$$

follows immediately. Therefore, satisfaction of (A.3) or, which is the same, $x\in\mathcal{E}(d_\alpha,(1-\xi_\alpha)^{-1}Q_\alpha)$ follows from conditions (A.1) and (A.2). For nondegeneracy of the ellipsoidal approximations, it suffices to require that $Q_\alpha=\alpha_0P_0+\sum\limits_{k=1}^{m}\alpha_kM_k>0$. Since the matrix $P_0>0$, there exist nonnegative numbers $\alpha_0,\ldots,\alpha_m$ such that $\sum\limits_{k=0}^{m}\alpha_k=1$ and $Q_\alpha>0$. Therefore, condition $Q_\alpha>0$ is constructive, which proves the theorem. □

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