LIMITING BEHAVIOUR OF BOUNDING ELLIPSOIDS FOR STATE ESTIMATION

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Abstract: Ellipsoidal technique is an advantageous and helpful tool in state estimation of dynamic systems with bounded disturbances; it gives a reliable outer approximations on reachable sets. In this paper the problem concerning asymptotic behaviour of ellipsoidal estimates is considered for linear discrete-time systems without measurements. At the core of that question, some benefits and useful properties were noted using the trace criterion in ellipsoidal calculus algorithms – specifically the boundedness and convergence of approximations. The minimal ellipsoid that contains the limiting reachable set is compared with the recursive algorithm of state estimation.

Keywords: asymptotic properties, bounded noise, discrete-time systems, ellipsoidal technique, reachable states, state estimation.

1. INTRODUCTION

Deterministic approach for parameter or state estimation of dynamic systems is rather extended and well developed as an effective alternative technique for stochastic Kalman filtering (Bertsekas, and Rhodes, 1971; Chernousko, 1988; Kurzhanskii, 1977; Kurzhanskii, and Valyi, 1996; Milanese, *et al.*, 1996; Schweppe, 1973). Such an approach implies that all system vectors and values are nonrandom but bounded in some sense. It is more natural in many state-space model descriptions. The main notion in this framework is a reachable set of dynamic system that is all possible states of the system subjected to unknown-but-bounded perturbations. The principal and useful tool, that provides a lot of benefits, is ellipsoidal one to describe reachable sets particularly for linear systems with additive disturbances. Ellipsoidal state outer-bounding algorithms were constructed in the works of Chernousko (1988), Durieu, *et al.* (2001), Fogel and Huang (1982), Schweppe (1973). They are based on two operations of ellipsoidal calculus: its summation and intersection, which correspond to the prediction and correction phase of the recursive state estimation (filtering) problem respectively. In this way, two basic ellipsoidal size measures are used to obtain an optimal approximation. The first one is volume (determinant) criterion and the second – trace criterion, which is the sum of squares of its semi-axes, see (Durieu, *et al.*, 2001).

The present paper analyzes asymptotic properties of ellipsoidal estimations for stable linear discrete-time dynamic systems without measurements. Chernousko (1980), Ovseevich and Reshetnyak (1992), Ovseevich (1994) investigated asymptotics of ellipsoids for continuous-time system. The local stability of system equilibrium points in some cases was shown. But general validation of global behaviour for optimal ellipsoids seems to be vague. For discrete-time case we operate with the explicit recursive matrix equations and asymptotic analysis becomes simpler. However this result will depend essentially on applied ellipsoidal size criterion when we construct outer-bounding approximations of reachable sets. The boundedness and some limiting properties of ellipsoids are proved with the trace criterion. In this context, illustrative examples are presented to show the asymptotic performance for both (volume and trace) criteria. The limiting ellipsoid of state estimation algorithm compares with the minimal ellipsoid containing the limiting reachable set.

2. NOTATION

Vector norm $\|x\|$, $x = (x_1, \ldots, x_n)^T \in R^n$ is understood below as Euclidean one: $\|x\|^2 = \sum x_i^2$. For symmetric matrices we write $P \geq 0$ if it is nonnegative definite and $P > 0$ – if it is only positive definite.

An ellipsoid in R^n is the set

$$E(c, P) = \{x : \left(P^{-1}(x - c), \ x - c\right) \leq 1\}. \quad (1)$$

$P \geq 0$ is an ellipsoidal matrix and $c \in R^n$ is its center. Eigenvalues of P are real and equal to squares of semi-axes. Therefore, $\det P$ and $\operatorname{tr} P$ are natural measures of size of the ellipsoid. If the ellipsoid considered has an empty interior, then the matrix P^{-1} in (1) is understood as pseudo-inverse.

Denote the spectral radius of matrix A as $\rho(A)$. Then, A is said to be stable iff $\rho(A) < 1$. The main limiting property for stable matrices is valid:

$$\lim_{k \to \infty} A^k = 0. \quad (2)$$

3. STATE ESTIMATION FOR DISCRETE-TIME SYSTEM

Consider a linear discrete-time dynamic system

$$x_{k+1} = Ax_k + Bw_k, \qquad k = 0, 1, 2, \ldots \quad (3)$$

$x_k \in R^n$ is a state vector of the model, $w_k \in R^m$ – an unknown disturbance vector, $A \in R^{n \times n}$ and $B \in R^{n \times m}$ – real known matrices. Suppose the disturbances w_k are non-random and bounded, i.e. $\|w_k\| \leq 1$. Ellipsoidal performance of disturbances is described by matrix B: $Bw_k \in E(0, BB^T)$, $\forall k$. The initial state vector x_0 of the system (3) is unknown, but we assume x_0 belongs to some given bounded ellipsoid $E_0 = E(0, P_0)$. Then,

$$D_k = \{x_k \in R^n : \\ x_{k+1} = Ax_k + Bw_k, \ x_0 \in E_0, \ \|w_k\| \leq 1\} \quad (4)$$

are reachable sets of the dynamic system (3). The sets D_k are compact and convex, as an algebraic sum of $k+1$ ellipsoids centered at the origin.

Lemma 3.1: The sequence of sets D_k $(k = 0, 1, 2, \ldots)$ converges iff the matrix A is stable.

Proof could be easily obtained if we write the support function for the set D_k.

Since the reachable set D_k is not an ellipsoid, it is proposed to find some outer-bounding ellipsoidal approximations and to construct the recursive equations for this estimates.

3.1 Problem Statement

As far as the sequence of D_k converges for stable linear systems (Lemma 3.1), the natural question arises concerning an asymptotic behaviour and properties of their ellipsoidal approximations. Here, it is proposed to consider the limiting behaviour of ellipsoidal state estimation for discrete-time dynamic system (3).

3.2 Recursive State Estimation

The summing of ellipsoids is the basic operation at the context of state estimation algorithms for dynamic system (3). Consider two bounded ellipsoids $E_1(c_1, P_1)$, $E_2(c_2, P_2)$ and its algebraic sum $S = E_1 + E_2 = \{x = x_1 + x_2 : \ x_1 \in E_1, \ x_2 \in E_2\}$.

Lemma 3.2 (Schweppe, 1973): Ellipsoid $E(c, P)$ with

$$c = c_1 + c_2, \quad P = \frac{P_1}{\gamma} + \frac{P_2}{1 - \gamma}, \quad 0 < \gamma < 1 \quad (5)$$

contains the sum S for all $\gamma \in (0, 1)$.

The minimal size ellipsoids are of our interest. The objective functions that correspond to volume and trace criteria are

$$f_1(P) = \ln \det P, \quad (6)$$

$$f_2(P) = \operatorname{tr} P. \quad (7)$$

Lemma 3.3 (Durieu, et al., 2001): The family $E(c, P(\gamma))$ from (5) contains the minimal ellipsoid which minimizes (6) or (7) among all ellipsoids in R^n approximating S.

Furthermore, ellipsoidal estimates of reachable sets can be constructed for linear dynamic system (3). We get the recursive algorithm (for arbitrary matrix $P_0 \geq 0$):

$$P_{k+1} = \frac{AP_kA^T}{\gamma_k} + \frac{BB^T}{1 - \gamma_k} \quad (8)$$

with

$$\gamma_k = \arg \min_{0 < \gamma_k < 1} f_1(P_{k+1}) \quad (9)$$

or

$$\gamma_k = \arg \min_{0<\gamma_k<1} f_2(P_{k+1}). \qquad (10)$$

The last two optimization problems for γ_k are convex on $0 < \gamma_k < 1$ (Durieu, *et al.*, 2001). The ellipsoid $E(0, P_k)$ contains the reachable set D_k for each $k = 1, 2, 3, \ldots$

3.3 Minimal Limiting Ellipsoid

Let matrix A to be stable. By definition, put

$$D_\infty = \lim_{k\to\infty} D_k. \qquad (11)$$

According to Lemma 3.1, D_∞ exists and it is closed and convex but in general it is not an ellipsoid. Then one should imbed it in some bounded ellipsoid

$$D_\infty \subset E(0, P). \qquad (12)$$

Note that the set D_∞ is centered at the origin. Any ellipsoid $E(0, P)$, $P > 0$, centered at zero and containing D_∞ satisfies the linear matrix inequality, see (Boyd, *et al.*, 1994):

$$P \geq \frac{APA^T}{\gamma} + \frac{BB^T}{1-\gamma} \qquad (13)$$

for some γ : $0 < \gamma < 1$. To find the minimal ellipsoid in terms of (6) or (7) we write the equality (Lyapunov equation)

$$P = \frac{APA^T}{\gamma} + \frac{BB^T}{1-\gamma} \qquad (14)$$

that has a unique positive definite solution iff $\rho^2(A) < \gamma < 1$. The one-parametric family $E(0, P(\gamma))$ from (14) ($\rho^2(A) < \gamma < 1$) contains the minimal ellipsoid, which minimizes $f_i(P(\gamma))$, $i = 1$ or 2 (see (6) and (7)) among all ellipsoids (12).

This assertion follows Boyd (1994). The two optimization problems

$$\gamma = \arg \min_{\rho^2(A)<\gamma<1} f_i(P(\gamma)), \quad i = 1, 2 \qquad (15)$$

are convex on interval $\rho^2(A) < \gamma < 1$. Therefore the minimal ellipsoid could be calculated by formulas (14) and (15), but no explicit solution is obtained.

For this reason, we seek some direct recursive method to achieve the minimal ellipsoid that contains D_∞. With the trace criterion this problem is reduced to minimization under constraints

$$\min_{\gamma,\, P(\gamma)} \text{tr}\, P(\gamma), \qquad (16)$$

where $\rho^2(A) < \gamma < 1$ and $P(\gamma)$ is determined from (14). If we consider Lagrangian function then the problem (16) is written as simultaneous equations

$$-\frac{\text{tr}\, YAPA^T}{\gamma^2} + \frac{\text{tr}\, YBB^T}{(1-\gamma)^2} = 0, \qquad (17)$$

$$Y = \frac{A^T Y A}{\gamma} + I, \qquad (18)$$

$$P = \frac{APA^T}{\gamma} + \frac{BB^T}{1-\gamma}, \qquad (19)$$

with new matrix variable $Y > 0$ (Lagrange multiplier). The equations could be solved via a recursive method

$$\gamma_{k+1} = \left(\frac{\text{tr}\, Y_k A P_k A^T}{\text{tr}\, Y_k A P_k A^T + \text{tr}\, Y_k B B^T} \right)^{1/2} \qquad (20)$$

with Y_k and P_k obtained from Lyapunov equations

$$Y_k = \frac{A^T Y_k A}{\gamma_k} + I, \qquad (21)$$

$$P_k = \frac{AP_kA^T}{\gamma_k} + \frac{BB^T}{1-\gamma_k}. \qquad (22)$$

This algorithm is stable for any initial γ_0 such that $\rho^2(A) < \gamma_0 < 1$ (when Lyapunov equation (21), (22) have positive definite solutions) and converges very quickly to the minimal limiting ellipsoid (see examples at section 5).

4. ASYMPTOTIC FOR MATRICES

4.1 Volume Criterion

Consider the recursive state estimation with volume (determinant) criterion. We have

$$P_{k+1} = \frac{AP_kA^T}{\gamma_k} + \frac{BB^T}{1-\gamma_k}, \qquad (23)$$

$$\gamma_k = \arg \min_{0<\gamma_k<1} \ln \det P_{k+1} \qquad (24)$$

Parameters γ_k could be obtained by solving a convex optimization problem (24). No explicit analytic solution is available for γ_k. Moreover, the recursive version of the algorithm leads to ill-conditioned (excessively elongated) ellipsoids. The next example illustrates the possible unbounded increase of approximations for a stable linear system.

Example 4.1: Consider matrix sequence (23), (24), where $n = 2$,

$$P_0 = I, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Matrix A is stable, i.e. $|a| < 1$; I is the identity matrix. Then we get the result for given data

$$P_k = \begin{pmatrix} 2 & 0 \\ 0 & (2a^2)^k \end{pmatrix}, \quad k = 1, 2, 3, \ldots$$

Matrix P_k tends to infinity as $k \to \infty$ when $1/\sqrt{2} < |a| < 1$. □

In this example the matrix pair A and B is not controllable (rank [B AB] = 1 < n = 2). But it is obvious that the above example can be generalized for controllable dynamic systems. Thus we can formulate the following assertion.

Proposition 4.1: The stability of linear systems does not guarantee the convergence or boundedness of state ellipsoidal approximations obtained with the volume criterion.

4.2 Trace Criterion

The use of trace criterion as the measure of ellipsoidal size makes easier the consideration of recursive matrix equations. The paper (Durieu, *et al.*, 2001) shows some advantages in this way with respect to its volume counterpart. In particular, since the trace of matrix is a linear function, an optimization problem (10) could be solved analytically for each k. Then, we rewrite the matrix sequence (8) in explicit form

$$P_{k+1} = \frac{\alpha_k + \beta}{\alpha_k} A P_k A^T + \frac{\alpha_k + \beta}{\beta} B B^T; \quad (25)$$
$$\alpha_k = (\operatorname{tr} A P_k A^T)^{1/2}, \quad \beta = (\operatorname{tr} B B^T)^{1/2}.$$

This algorithm makes it possible to obtain better conditioned (less elongated) ellipsoids for large k then with the determinant criterion.

Lemma 4.1: The necessary and sufficient condition for the matrix sequence (25) to be bounded is that $\rho(A) < 1$.

The necessity follows directly from Lemma 3.1 on the convergence of reachable sets. The sufficient condition has been proved in the paper (Nazin, 2001). The last lemma provides the convergence of some matrix subsequence in (25).

Further, in order to find an asymptotic ellipsoid of the recursive algorithm for the trace criterion, the limiting matrix equation should be considered. An equilibrium point $P \in R^{n \times n}$ of (25) is the solution of nonlinear equation

$$P = \frac{\alpha(P) + \beta}{\alpha(P)} A P A^T + \frac{\alpha(P) + \beta}{\beta} B B^T, \quad (26)$$

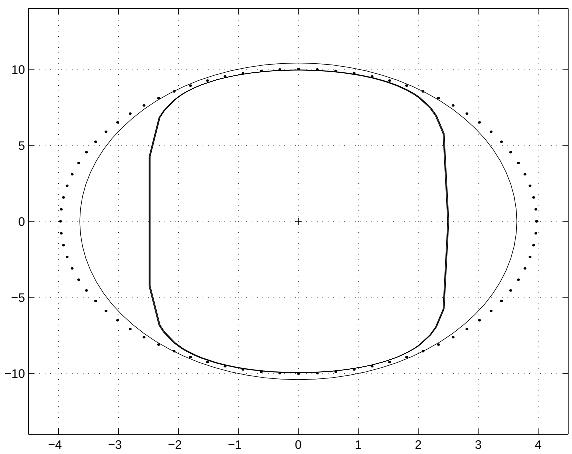

Fig. 1: Limiting ellipsoids for Example 1.

where $\alpha(P) = (\operatorname{tr} A P A^T)^{1/2}$.

Lemma 4.2: The matrix equation (26) has a unique positive definite solution.

The proof can be obtained if we write the matrix equation (26) in terms of Kronecker product (Horn and Johnson, 1985), then the solution can be written explicitly in vector form.

From lemmas 4.1 and 4.2 we conjecture that the entire matrix sequence (25) converges to the solution of equation (26). Nevertheless, this solution (ellipsoid $E(0, P)$) is not optimal for limiting reachable set D_∞. Indeed, the optimal ellipsoid is obtained with (14), (15), but the equation (26) differs from (14), (15). However this difference could be neglected in some cases. Examples below describe a few situations of asymptotic processes.

5. EXAMPLES

In this section two different examples are considered in order to illustrate an asymptotic properties of state estimation with trace criterion for stable linear dynamic systems.

Example 5.1: Let $n = 2$ and

$$A = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad B = I, \quad P_0 = I.$$

Then the reachable sets D_k and ellipsoidal estimates converge as far as matrix A is stable. Figure 1 shows the result of simulation. Here, the limiting reachable set D_∞ (solid bold line) and two ellipsoids are shown, namely the minimal limiting ellipsoid from (14) – pointed line, and the asymptotic ellipsoid of recursive process (25) – solid line. The minimal limiting ellipsoid is obtained from

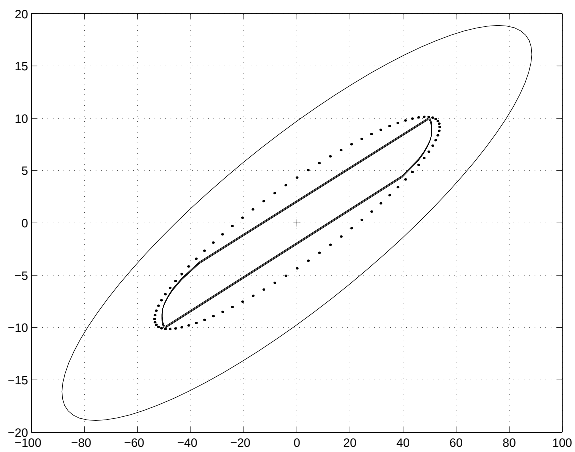

Fig. 2: Limiting ellipsoids for Example 2.

iterative method (20)–(22). Few iterations (less than 10) are needed to get a close approximation of D_∞.

Notice that the difference between two ellipsoids is rather small and the recursive algorithm gives a good estimation for D_∞. The reason is that matrix A of the system is symmetric and diagonal. However this difference increases when A becomes non-symmetric. □

Example 5.2: Consider $n = 2$ and

$$A = \begin{pmatrix} 0.6 & 2 \\ 0 & 0.9 \end{pmatrix}, \quad B = I, \quad P_0 = I.$$

In this case, ellipsoids are illustrated at figure 2. The difference between the minimal limiting ellipsoid (from (20)–(22)) and limiting ellipsoid of recursive algorithm (25) is essential. It is getting larger if we evaluate the non-diagonal element in matrix A. But definitely this ellipsoid will not become unbounded. □

6. CONCLUSION

Recursive ellipsoidal state estimation is an attractive algorithm of reachable sets approximation for dynamic systems. But ellipsoids may become unbounded even when the system is stable. In this way, some asymptotic aspects of such an approximations have been considered. The bounded recursive behaviour and limiting properties of ellipsoids were shown for the trace criterion that gives another of its benefits. Therefore the sum of squares of semi-axes as the ellipsoidal size measure are preferable for good asymptotic performance.

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