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DISCRETE SYSTEMS

Ellipsoidal State Estimates of Linear Dynamic Systems: Their Limiting Behavior[1]

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Abstract—In guaranteed estimation of the states of dynamic systems under deterministic bounded disturbances, use of made of outer ellipsoidal estimates that are optimal relative to the volume and sum of squares of the semiaxes of the ellipsoid. The boundedness and convergence of a sequence of such ellipsoids for stable discrete systems are investigated. The asymptotic behavior of these estimates mostly depends on the minimality criterion chosen for the ellipsoid dimension.

1. INTRODUCTION

Let us study the discrete-time linear stationary dynamic system described by the vector equation

$$x_{k+1} = Ax_k + Bw_k, \quad k = 0, 1, 2, \dots . \tag{1}$$

Here $x_k \in R^n$ is the phase vector of the system, $w_k \in R^m$ is a perturbation vector, and $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are real matrices of suitable dimension. We assume that external disturbances are deterministic but bounded, i.e., the vectors w_k are unknown at any instant, but $\|w_k\| \le 1$, where $\|\cdot\|$ is the Euclidean norm in the vector space. In such a deterministic approach, the main interest consists of describing and constructing the reachability set for the system at instant k under certain initial conditions. But this set may prove rather complicated in structure. Therefore, in the general case, a simple and obvious approach, first developed in [1], is the approximation of the reachability set by a class of domains of specific canonical form, in particular, the class of ellipsoids. Obviously, these ellipsoids must be the least ellipsoids in some sense. The volume and the sum of squares of the lengths of the semiaxes of the ellipsoid are used as an optimality criteria.

Let $E(c, P) = \{x \in R^n : (x - c)^T P^{-1}(x - c) \le 1\}$ be an ellipsoid in the space R^n, where c is the center of the ellipsoid and P is a symmetric positive-definite matrix.

Let the vector x_0 at the initial instant belong to the ellipsoid $E_0 = E(c_0, P_0)$. Then the reachability set $D_k = \{x_k \in R^n : x_0 \in E(c_0, P_0)\}$, $k = 0, 1, 2, \dots$, of the linear system (1) is compact and convex [2]. In [2, 3], guaranteed estimates for this ellipsoid in [2, 3] are constructed via ellipsoidal approximation of the sum of two ellipsoids. The evolution of the parameters of the ellipsoid $E_k(c_k, P_k)$ describing the state of system (1) is defined by the equation

$$c_{k+1} = Ac_k, \quad P_{k+1} = \frac{AP_kA^T}{\gamma_k} + \frac{BB^T}{1-\gamma_k}, \quad 0 < \gamma_k < 1. \tag{2}$$

The parameter γ_k depends on the minimality criterion chosen for the resultant ellipsoid. In particular [2, 3],

$$\gamma_k = \arg\min_{0<\gamma_k<1} \operatorname{Det} P_{k+1} \tag{3}$$

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for the minimal volume of the ellipsoid, or

$$\gamma_k = \arg \min_{0<\gamma_k<1} \operatorname{Tr} P_{k+1} \tag{4}$$

for the minimal sum of squares of the lengths of semiaxes. The minimization problems (3) and (4) have unique solutions. According to [4], other optimality criteria are in general better suited for ellipsoidal estimation.

In studying the reachability sets of system (1) for $x_0 \in E(c_0, P_0)$, we must recall that a sequence of reachability sets D_k converges if and only if the matrix A of the system is stable, i.e., only if its spectral radius $\rho(A) \equiv |\lambda_i(A)|_{\max} < 1$. Then the limiting reachability set D_∞ can be described exactly, for example, with the support function apparatus. The obvious question that arises here is the asymptotic behavior of locally optimal ellipsoidal estimates of the reachability domains of stable systems. Note that for $\rho(A) \geq 1$, the sequence of ellipsoids is unbounded due to the divergence of the sequence $\{D_k\}$.

Such a problem for continuous linear dynamic systems is investigated in [2, 5, 6] concerned with the equilibrium points of differential equations for locally optimal ellipsoids of stable systems (in [2], only a variant of diagonal matrices of a system is studied). A concept of local optimality of the ellipsoid $E = E(t)$ is applied for minimizing the growth rate of some functional $L = L(E)$ at an arbitrary instant t, i.e., $\dfrac{d}{dt}L(E(t)) \to \min$. The cases $L = \operatorname{Tr}(CQ)$ and $L = \operatorname{Det} Q$, where Q is the matrix of the ellipsoid and C is a fixed positive-definite matrix, are considered. In particular, it is asserted that the equilibrium points are unique and locally stable if $L(Q) = \operatorname{Tr}(CQ)$. If $L(Q) = \operatorname{Det} Q$, this assertion holds under more stringent conditions. Nevertheless, the general global behavior of optimal ellipsoids for continuous stable systems is rather uncertain. For example, it is not clear whether these systems would converge or be bounded.

In this paper, we study the limiting behavior of ellipsoids as $k \to \infty$ for a stable linear discrete dynamic system, using two criteria, viz., minimal volume (determinant of the matrix of the ellipsoid) and minimal sum of the squares of the lengths of semiaxes (trace of the matrix of the ellipsoid).

2. APPROXIMATION BY THE DETERMINANT CRITERION

Let us consider the recurrent Eqs. (2) for the parameters of the ellipsoid $E_k = E(c_k, P_k)$ describing the state of the dynamic system (1) in terms of the minimal volume of the ellipsoid [2, 3]:

$$c_{k+1} = Ac_k, \quad P_{k+1} = \frac{AP_kA^T}{\gamma_k} + \frac{BB^T}{1-\gamma_k}; \quad \gamma_k = \arg \min_{0<\gamma_k<1} \operatorname{Det} P_{k+1}. \tag{5}$$

In this approach, the ellipsoid may be rather flat, thereby may create difficulties if k is large.

Let the matrix A of the system be stable, i.e., its spectral radius $\rho(A) < 1$. Then the sequence of centers c_k in (2) and (5) converges to zero [7]:

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} A^k c_0 = 0. \tag{6}$$

Therefore, to study the limiting behavior of ellipsoids, we may restrict ourselves only to the equation of their matrices. But it is not easy to analyze this equation due to its implicit form.

Nevertheless, the following example shows that the sequence of ellipsoids (5) may be unbounded even for a stable linear system.

Example 1. Let us consider sequence (5) for the two-dimensional case under the conditions

$$c_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_0 = I, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $0 < a < 1$ for the stability of the matrix A. Here I is a unit matrix.

Then all centers c_k are zero. Applying the recurrent formula (5) for matrices, we obtain a simple sequence of ellipsoids $E_k = E(c_k, P_k)$:

$$c_k = c_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P_k = \begin{pmatrix} 2 & 0 \\ 0 & 2^k a^{2k} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & (2a^2)^k \end{pmatrix}.$$

Let $\widetilde{P} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\widehat{P} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$P_k \to \begin{cases} \widetilde{P}, & 0 < a < 1/\sqrt{2} \\ \widehat{P}, & a = 1/\sqrt{2} \\ \infty, & 1/\sqrt{2} < a < 1. \end{cases}$$

Thus, the sequence of least-volume ellipsoids in this example is unbounded (for $1/\sqrt{2} < a < 1$), though the matrix A of system (1) is stable.

Note. Under the condition of example 1, the matrix pair (A, B) is noncontrollable, because $\operatorname{rank}[B\ AB] = 1 < n = 2$. But the result can also be applied to a controllable system, taking

$$A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

Here ε is an arbitrary real number. We can show that, for any ε, ellipsoids with parameters (5) also diverge for some $a \in (0, 1)$.

An example of a unbounded bounding ellipsoid of minimal volume for continuous systems is given in [2]. The following assertion summarizes our results.

Assertion. *The stability of the matrix A of a linear system does not guarantee the convergence and boundedness of ellipsoidal estimates computed by formula* (5) *under the determinant criterion.*

For the existence and determination of the limiting ellipsoid in this case, it is necessary that the initial stable system satisfy much stronger sufficiency conditions.

3. APPROXIMATION BY THE TRACE CRITERION

The sum of squares of the semiaxes of an ellipsoid, i.e., the trace of the corresponding matrix, is the most convenient criterion, because the trace is a linear function of the matrix. Therefore, ellipsoids are not as flat as under the determinant criterion and their parameters can be defined by explicit formulas [3, 8].

The parameter γ_k in the recurrent Eqs. (2) is easily computed from condition (4):

$$\gamma_k = \frac{\alpha_k}{\alpha_k + \beta}, \quad \text{where} \quad \alpha_k = \sqrt{\operatorname{Tr}(AP_kA^T)}; \quad \beta = \sqrt{\operatorname{Tr}(BB^T)}. \tag{7}$$

Therefore, for the ellipsoids $E_k = E(c_k, P_k)$ obtained under the trace criterion, we have

$$c_{k+1} = Ac_k, \quad P_{k+1} = \frac{\alpha_k + \beta}{\alpha_k} AP_kA^T + \frac{\alpha_k + \beta}{\beta} BB^T, \tag{8}$$
$$\alpha_k = \sqrt{\mathrm{Tr}\,(AP_kA^T)}, \quad \beta = \sqrt{\mathrm{Tr}\,(BB^T)} = \mathrm{const}.$$

As before, we assume that the matrix A of the system is stable, i.e.,

$$\rho(A) < 1. \tag{9}$$

We now formulate the main result.

Theorem. *If condition* (9) *holds, i.e., the linear system* (1) *is stable, then the sequence of ellipsoids computed using the trace criterion* (4) *is bounded.*

The proof of the theorem is given in the Appendix.

The boundedness of approximations by minimal-trace exterior ellipsoids for stable systems implies the convergence of some subsequence of ellipsoids (8). The convergence of the whole sequence of ellipsoids (8) requires a detailed analysis of nonlinear, recurrent, and stationary matrix equations. This topic will be discussed in a future publication.

We now give an example to illustrate the theorem for a stable matrix A.

Example 2. Let us consider a linear dynamic system under the conditions of example 1, in particular, $n = 2$, $c_0 = 0$, $P_0 = I$, $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Assuming that the matrix A is stable, i.e., $0 < a < 1$, we find that all centers c_k of ellipsoids are zero. Applying formula (8) to the matrices, we obtain

$$P_k = \begin{pmatrix} \alpha_k + 1 & 0 \\ 0 & \alpha_k(\alpha_k + 1) \end{pmatrix}, \quad \alpha_{k+1} = a\sqrt{\alpha_k(\alpha_k + 1)}, \quad \alpha_0 = a.$$

Since the numerical sequence $\alpha_{k+1} = a\sqrt{\alpha_k(\alpha_k + 1)}$, $\alpha_0 = a$, is bounded and monotonic, it converges and

$$\alpha_k \to \alpha = \begin{cases} 0, & a < a^* \\ a^*, & a = a^* \\ a^2/(1 - a^2), & a > a^*, \end{cases}$$

where $a^* = (-1 + \sqrt{5})/2 \approx 0.618$ is the root of the equation $a^2 + a - 1 = 0$.

Consequently, the matrices P_k, and along with them the sequence of ellipsoids, are bounded and converge.

The formulas for ellipsoids in examples 1 and 2 are very simple, because the matrices A and B are degenerate. But in more general cases, computations are rather cumbersome.

Example 3. Let $A = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $c_0 = 0$, and $P_0 = I$.

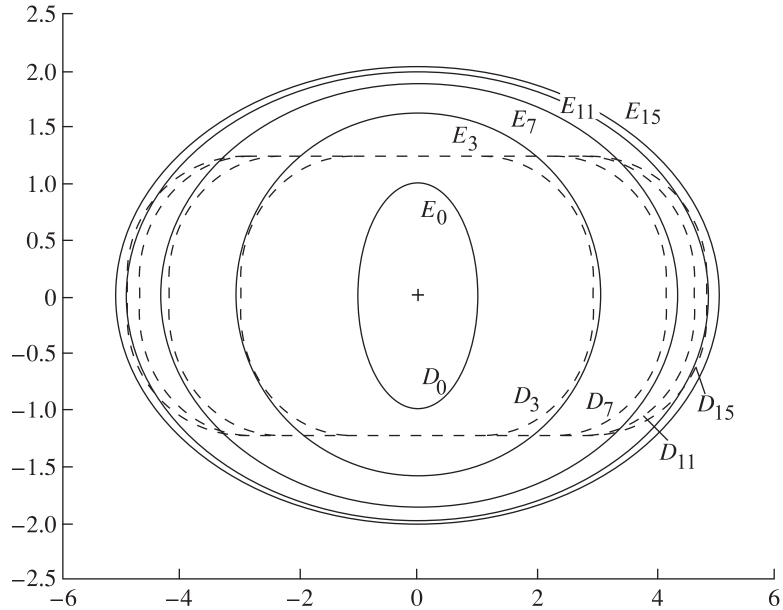

Figure.

Figure shows the reachability domains D_k (broken curves) and their bounding ellipsoids E_k (solid curves) for $k = 0, 3, 7, 11$, and 15 for this example. Clearly, the sequence $E_k = E(c_k, P_k)$ converges as $k \to \infty$. Moreover, the limiting ellipsoid, by formula (8), is $E_\infty = E(c_\infty, P_\infty)$, where

$$c_\infty = 0, \quad P_\infty \approx \begin{pmatrix} 27.4264 & 0 \\ 0 & 4.2012 \end{pmatrix}.$$

4. CONCLUSIONS

The sequence of ellipsoids with minimal-trace matrices for linear stable discrete systems is bounded. This is an important advantage of these ellipsoids compared to the ellipsoids found from the widely used minimal-volume criterion. As demonstrated by examples 2 and 3, the sequence of ellipsoids converges for the matrix trace in all cases. But thus far it has not been possible to prove this assertion and to find the limiting ellipsoid.

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APPENDIX

Proof of Theorem. It suffices to show that the sequence of matrices P_k in (8) is bounded, because the centers of the ellipsoids $E(c_k, P_k)$, according to (6), all converge to zero.

Without loss of generality, we assume that the matrices A and B are nonzero.

Consider the equation

$$P_{k+1} = AP_kA^T + \frac{\beta}{\alpha_k}AP_kA^T + \frac{\alpha_k}{\beta}BB^T + BB^T.$$

For matrix norms, we have

$$\|P_{k+1}\| \leq \|A\|\,\|P_k\|\,\|A^T\| + \frac{\beta}{\alpha_k}\|AP_kA^T\| + \frac{\alpha_k}{\beta}\|BB^T\| + \|BB^T\|,$$
$$\|P_{k+1}\| \leq \|A\|\,\|P_k\|\,\|A^T\| + \alpha_k\left(\beta\frac{\|AP_kA^T\|}{\alpha_k^2} + \frac{\|BB^T\|}{\beta}\right) + \|BB^T\|.$$

We now state a few simple assertions.

Assertion 1. *For every given matrix A and number $\varepsilon > 0$, there exists at least one matrix norm $\|\cdot\|_\varepsilon$ such that*

$$\begin{aligned}\rho(A) \leq \|A\|_\varepsilon \leq \rho(A) + \varepsilon = \rho_\varepsilon(A) < 1,\\ \rho(A) \leq \|A^T\|_\varepsilon \leq \rho(A) + \varepsilon = \rho_\varepsilon(A) < 1.\end{aligned} \tag{A.1}$$

The proof of Assertion 1 is given in [7].

Assertion 2. *If P is a symmetric positive-definite matrix and $\alpha(P) = \sqrt{\mathrm{Tr}\,(APA^T)}$, then there exists a number such that*

$$\frac{\|APA^T\|}{\alpha^2(P)} \leq C$$

for any matrix norm.

Proof. Since the matrix APA^T is positive definite, we have $\mathrm{Tr}\,(APA^T) = \sum_{i=1}^{n} \lambda_i(APA^T) \geq \rho(APA^T) > 0$, where $\rho(APA^T)$ is the spectral radius of the corresponding matrix. Its operator norm is

$$\|APA^T\| = \max_{\|x\|=1} \|APA^Tx\| = \rho(APA^T) \leq \mathrm{Tr}\,(APA^T).$$

Therefore, $\|APA^T\|/\alpha^2(P) = \|APA^T\|/\mathrm{Tr}\,(APA^T) \leq 1$, from which we easily obtain the inequality for the operator norm and, consequently, for any other matrix norm.

Assertion 3. *For any symmetric positive-definite matrix $P > 0$, we have $\alpha(P) \leq C\|P\|_\varepsilon^{1/2}$, where $\alpha(P) = \sqrt{\mathrm{Tr}\,(APA^T)}$ and $C = \mathrm{const} > 0$.*

Proof. Let $\rho(P)$ be the spectral radius of the $n \times n$ matrix $P > 0$. Now, by Assertion 1, we obtain $\mathrm{Tr}\,P \leq n\,\rho(P) \leq n\|P\|_\varepsilon$ and

$$\alpha^2(P) = \mathrm{Tr}\,(APA^T) \leq |\lambda_i(A^TA)|_{\max}\mathrm{Tr}\,P \leq |\lambda_i(A^TA)|_{\max} n\|P\|_\varepsilon = C\|P\|_\varepsilon.$$

Since $\alpha(P) \geq 0$, we have $\alpha(P) \leq C\|P\|_\varepsilon^{1/2}$ and $C = \mathrm{const} > 0$.

By Assertions 1 and 2, we obtain

$$\begin{aligned}\|P_{k+1}\|_\varepsilon \leq \rho_\varepsilon^2(A)\|P_k\|_\varepsilon + \alpha_k\left(\beta C + \frac{\|BB^T\|_\varepsilon}{\beta}\right) + \|BB^T\|_\varepsilon,\\ \|P_{k+1}\|_\varepsilon \leq \rho_\varepsilon^2(A)\|P_k\|_\varepsilon + C_1\alpha_k + C_2.\end{aligned} \tag{A.2}$$

Applying Assertion 3 to inequality (A.2), we obtain

$$\|P_{k+1}\|_\varepsilon \le \rho_\varepsilon^2(A)\|P_k\|_\varepsilon + \widetilde{C}_1\|P_k\|_\varepsilon^{1/2} + C_2.$$

Since $2\|P_k\|_\varepsilon^{1/2} \le \frac{1}{M}\|P_k\|_\varepsilon + M$ for any $M > 0$, we have

$$\|P_{k+1}\|_\varepsilon \le \left(\rho_\varepsilon^2(A) + \frac{\widetilde{C}_1}{2M}\right)\|P_k\|_\varepsilon + \widetilde{C}_2. \qquad \text{(A.3)}$$

According to (A.1), $\rho_\varepsilon(A) < 1$. Therefore, there exists an $M_0 > 0$ such that

$$\rho_\varepsilon^2(A) + \frac{\widetilde{C}_1}{2M} < 1 \ \forall M > M_0.$$

Therefore, (A.3) implies that

$$\|P_k\|_\varepsilon \le \frac{\widetilde{C}_2}{1 - (\rho_\varepsilon^2(A) + \widetilde{C}_1/2M)} \ \forall k = 0, 1, 2, \ldots.$$

Hence the sequence of matrices P_k is bounded and, consequently, the sequence of ellipsoids is also bounded. This completes the proof.

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