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Interval solutions for interval algebraic equations

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Abstract

In the framework of interval uncertainty, a well-known classical problem in numerical analysis is considered, namely, to find "the best" interval solution for interval system of linear algebraic equations. This problem is known to be NP-hard and can be solved via multiple linear programming. In present paper, a simple approach is proposed for some particular models of interval uncertainty. This method gives an optimal interval solution without linear programming and is tractable for moderate-size problems. For large-scale problems an effective overbounding technique is developed.

Keywords: Interval bounding; Interval solution; Linear interval equations; Solution set

1. Introduction

Solution of linear interval system of algebraic equations is a challenging problem in interval analysis and robust linear algebra. This problem was first considered at the middle of 1960s by Oettli and Prager [10] and was pointed out as very important for numerous applications. Since that, this problem has received much attention and was developed in the context of modeling of uncertain systems (see [1–16]).

Consider a system of linear algebraic equations

$$\boldsymbol{A}x = \boldsymbol{b} \tag{1}$$

with $x \in \mathbb{R}^n$, interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and interval vector $\boldsymbol{b} \in \mathbb{IR}^n$. The matrix and vector are said to belong to interval family if their elements are from some real intervals $[a; b]$, $a \leq b$. Here the standard notations $\mathbb{IR}^{n\times n}$ and $\mathbb{IR}^n$ are used for sets of all n-dimensional interval square matrices and vectors, respectively. System (1) is called the interval system of equations. Suppose $\boldsymbol{A}$ is regular, i.e. A is nonsingular for any $A \in \boldsymbol{A}$. Then for a matrix $A \in \boldsymbol{A}$ and any vector $b \in \boldsymbol{b}$ an ordinary linear system $Ax = b$ has the unique solution. We are interested in a set X of all these solutions of interval system:

$$X = \{x \in \mathbb{R}^n : Ax = b, \quad A \in \boldsymbol{A}, \quad b \in \boldsymbol{b}\}. \tag{2}$$

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The characterization of this set has been obtained in [10]. It has been proved that the intersection of X with each orthant in $\mathbb{R}^n$ gives a convex polytope. But in general, X is non-convex as the union of convex sets, and its detailed description meets combinatorial difficulties. The main objective is to find interval solution of linear interval system that is to determine the smallest interval vector $\boldsymbol{X}^*$ containing all possible solutions. In other words, we need to imbed the solution set X into the minimal box in $\mathbb{R}^n$. This problem is known to be NP-hard [7] and complicated from computational viewpoint for large-scale systems. Oettli [11] shows how multiple linear programming can be used to obtain $\boldsymbol{X}^*$; this line of research was continued by Cope and Rust [2], and Rust and Burrus [15]. Some iterative approaches were established at this context as well as direct numerical methods that provide overbounding of $\boldsymbol{X}^*$ (see monographs [5,6,9] and papers [14,16]).

In this paper we present a simple approach for interval approximation of the solution set. Instead of solving linear programming in each orthant it is proposed to deal with scalar equation. This method is based on Rohn's result [13] and simplifies his algorithm. To find the interval estimates of the solution set X, all vertices of the convex hull of X should be obtained. The search of the each vertex is reduced to the solution of scalar equation. The technique described is easily extended to more general types of uncertainty such as structured perturbations and linear fractional representation of uncertainty. We also provide simple and fast procedures for overbounding interval solutions.

2. Problem statement

Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ be a nominal data and consider a family of perturbed systems of n equations with n unknowns

$$(A + \Delta A)x = b + \Delta b, \tag{3}$$

where matrix uncertainty $\|\Delta A\|_\infty \le \varepsilon$ and right-hand side vector uncertainty $\|\Delta b\|_\infty \le \delta$ are bounded in ∞-norm. The infinity matrix or vector norm is equal to maximal absolute value of its elements. The nominal real matrix A and real vector b are known. Matrix A is assumed to be nonsingular. This is a particular case of interval uncertainty for linear system of algebraic equations when $-\varepsilon \le (\Delta A)_{ij} \le \varepsilon$ and $-\delta \le (\Delta b)_i \le \delta$ for all $i, j = 1, \dots, n$ (more general models of uncertainty are addressed in Section 4.3). Then the exact solution set

$$X = \{ x \in \mathbb{R}^n : (A + \Delta A)x = b + \Delta b, \quad \|\Delta A\|_\infty \le \varepsilon, \quad \|\Delta b\|_\infty \le \delta\} \tag{4}$$

illustrates all possible system solutions under given constraints. The problem is to describe explicitly the set $X \ni x$ (Section 3) and to derive its optimal interval outer-bounding estimate $\boldsymbol{X}^*$ (Section 4). This set is of the form $\boldsymbol{X}^* = \{x : \underline{x}_k \le x_k \le \bar{x}_k\}$, where

$$\underline{x}_k = \min\nolimits_{x\in X} x_k, \qquad \bar{x}_k = \max\nolimits_{x\in X} x_k. \tag{5}$$

More tractable problem is to find any "good" overbounding of $\boldsymbol{X}^*$ that often can be written in explicit analytic form (Section 5), i.e. an interval set $\boldsymbol{X}$ containing $\boldsymbol{X}^* : \boldsymbol{X} \supseteq \boldsymbol{X}^*$.

3. Solution set

The detailed description of the solution set for linear interval systems was given in the pioneer work by Oettli and Prager [10] for general situation of interval uncertainty. In our case their result is reduced as follows.

Lemma 1 (Oettli and Prager [10]). *The set of all admissible solutions of system* (3) *is a polytope*:

$$X = \{x : \|Ax - b\|_\infty \le \varepsilon \|x\|_1 + \delta\}. \tag{6}$$

Here $\|x\|_1 = \sum |x_k|$. Generally X is a non-convex set. If X lies in a given orthant of $\mathbb{R}^n$, the right-hand side of the inequality in (6) is a linear function. Therefore, X becomes convex in this case. And its interval approximation leads to convex optimization (5). However this is no longer the case in most situations. The following example illustrates the structure of the solution set.

Example 1. Consider the system (3) with $A = \mathrm{diag}\{1, 1\}$, $b = (1, 1)^{\mathrm{T}}$, $\varepsilon = 0.3$ and $\delta = 0$. The exact solution of the nominal linear system is the point $x^* = A^{-1}b = (1, 1)^{\mathrm{T}}$. Then the solution set

$$X = \{x = (x_1,\ x_2)^{\mathrm{T}} : \max\{\,|x_1 - 1|,\ |x_2 - 1|\,\} \le 0.3(|x|_1 + |x_2|)\} \tag{7}$$

lies in the first orthant on two-dimensional plane. It is bounded and convex for $\varepsilon < 0.5$ (see Fig. 1a) . But if we take $A = \begin{pmatrix} 1 & 0.4 \\ 0 & 1 \end{pmatrix}$, $b = (1, 0)^{\mathrm{T}}$, then X becomes non-convex for any $\varepsilon > 0$, that is shown in Fig. 1b.

3.1. Nonsingularity radius

For a nonsingular matrix A there exists the margin of perturbations ΔA, which preserve nonsingularity of the matrix $A + \Delta A$ and boundedness of the solution set for the interval system (3). More precisely, we define nonsingularity radius for ∞-norm as follows:

$$\rho(A) = \inf\{\varepsilon : A + \Delta A \text{ is singular for some } \|\Delta A\|_\infty \le \varepsilon\} = \sup\{\,\varepsilon : \boldsymbol{A} \text{ is regular}\}. \tag{8}$$

Recall that $(\infty, 1)$ matrix norm is defined as

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty \le 1} \sum_{i=1}^{n} |a_i^{\mathrm{T}} x|. \tag{9}$$

Vector a_i denotes i-th row of matrix A.

Lemma 2 ([13] (see also [6,9,12,14]).) *Nonsingularity radius for interval perturbations is reciprocal to the* $(\infty, 1)$*-norm of the inverse matrix*

$$\rho(A) = \frac{1}{\|A^{-1}\|_{\infty,1}}. \tag{10}$$

Calculation of such norm is NP-hard problem [1]. It suffices to compute 2^n numbers that is acceptable for moderate n, say $n \le 15$. For large-scale systems there exist some effective methods to approximate this value [1,8].

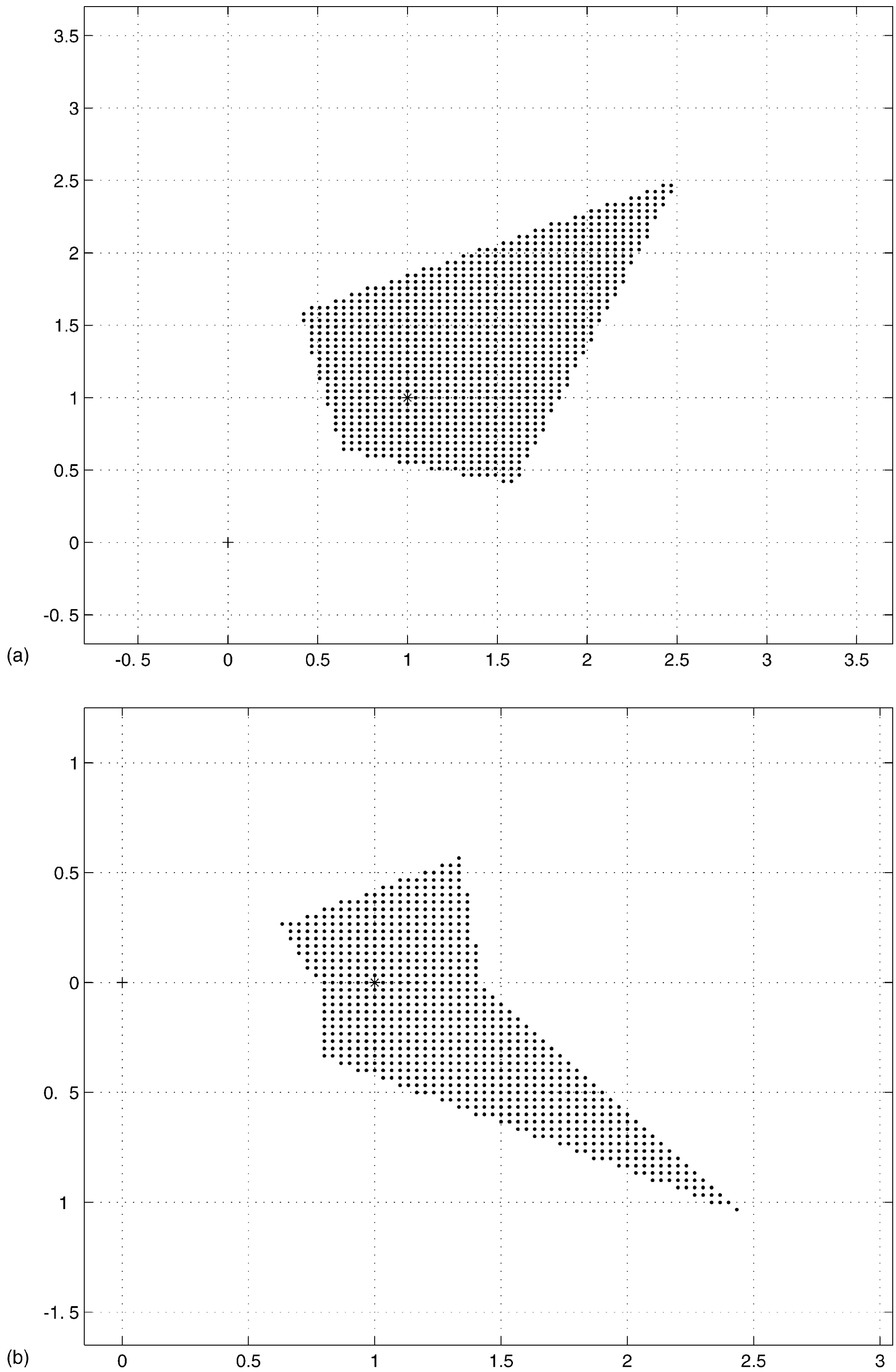

Fig. 1. (a) Solution set (*variant* 1). (b) Solution set (*variant* 2).

4. Optimal interval estimates of the solution set

The problem is to determine lower $\underline{x}_i$ and upper $\bar{x}_i$ bounds on each components x_i of vector $x \in \mathbb{R}^n$ under assumption $x \in X$. In this section we provide a new simple approach for interval estimation by searching for vertices of the convex hull *ConvX* of the solution set X instead of solving linear programming in each orthant. The main base of this technique is the paper by Rohn [13], where a key result defining *ConvX* was proved.

Consider the inequality that defines the solution set

$$\|Ax - b\|_\infty \le \varepsilon \|x\|_1 + \delta. \tag{11}$$

This scalar inequality is equivalent to n inequalities

$$\begin{aligned} |a_1^{\mathrm{T}} x - b_1| &\le \varepsilon \|x\|_1 + \delta, \\ &\vdots \\ |a_n^{\mathrm{T}} x - b_n| &\le \varepsilon \|x\|_1 + \delta. \end{aligned} \tag{12}$$

Instead of inequality in (12) let us write the equality

$$|a_i^{\mathrm{T}} x - b_i| = \varepsilon \|x\|_1 + \delta, \quad i = 1, \dots, n. \tag{13}$$

Introduce a set of vectors $s \in \mathbb{R}^n$, $S = \{s : |s_i| = 1\}$, and consider a system of equations for some $s \in S$

$$(a_i^{\mathrm{T}} x - b_i) s_i = \varepsilon \|x\|_1 + \delta, \quad i = 1, \dots, n. \tag{14}$$

Lemma 3 (Rohn [13]). *For given nominal matrix A, let an interval family*

$$\boldsymbol{A} = \{A + \Delta A : \|\Delta A\|_\infty \le \varepsilon\} \tag{15}$$

be regular that is all matrices in $\boldsymbol{A}$ are nonsingular. Then each vertex of ConvX satisfies the nonlinear system of equations (14), *which has an exactly one solution for every fixed vector $s \in S$.*

4.1. Main result

To simplify the analysis let $y = Ax - b$. After changing the variables the equalities in (14) are converted to

$$y_i = (\varepsilon \|A^{-1}(y + b)\|_1 + \delta)/s_i, \quad i = 1, \dots, n. \tag{16}$$

Recall that $s_i = \pm 1\ \forall i$. The transformed solution set $Y = \{y : (16)\ \text{holds}\}$ is then the linear image of X that is $Y = AX - b$. Note that $ConvY = A\ \ ConvX - b$. For any $\varepsilon > 0$ the set Y lies in every orthant in $\mathbb{R}^n$, i.e. the intersection of Y with each orthant is not empty. Moreover, each orthant contains some number of vertices of *ConvY*. Vector $s = (s_1, \dots, s_n)^{\mathrm{T}} = \operatorname{sign} y$ specifies the choice of orthant under consideration. Number of different vectors $s \in S$ is 2^n and is equal to the number of orthants. According to lemma 3, if $s \in S$ is fixed, then the only solution of equation (16) gives a unique vertex of convex hull *ConvY* in given orthant. Taking all various vectors s we find all vertices of *ConvY*.

Therefore let $s = s^0$. Then the equality (16) leads to one scalar equation

$$\tau = \varphi(\tau), \tag{17}$$

where $\tau = |y_i|$, $y = \tau s^0$ and $\varphi(\tau) = \varepsilon \|A^{-1}(\tau s^0 + b)\|_1 + \delta$. Denote $G = A^{-1}$ and g_i, i-th row of matrix G, then

$$\varphi(\tau) = \varepsilon \sum_{i=1}^{n} |(g_i, s^0)\tau + x_i^*| + \delta, \tag{18}$$

where $x^* = A^{-1}b$. Function $\varphi(\tau)$ is defined for $\tau \geq 0$ and is a convex piece-wise linear function of τ.

Lemma 4. *For regular interval family* $\boldsymbol{A}$ *scalar equation* (17) *has a unique solution over* $(0, \infty)$.

Proof. The initial value $\varphi(0) = \varepsilon\|x^*\|_1 + \delta > 0$. Since $\boldsymbol{A}$ is regular, then $\varepsilon < \rho(A) = 1/\|G\|_{\infty,1}$. It means that $\|G\|_{\infty,1} = \max_{s\in S} \sum_{i=1}^{n} |(g_i, s)| < 1/\varepsilon$ and the derivative in infinity is then

$$\varphi'(\infty) = \varepsilon \sum_{i=1}^{n} |(g_i, s^0)| < 1. \tag{19}$$

As long as function $\varphi(\tau)$ is convex, then $\varphi'(\tau) < 1$ for all $\tau > 0$ and the equation (17) has the only one solution $\tau^* > 0$. □

The solution τ^* can be obtained using a simple iterative scheme, for example, Newton iterations. Consider $\tau_k \geq 0$ and linearized equation $\tau = \varphi(\tau_k) + \varphi'(\tau_k)(\tau - \tau_k)$; take its solution as τ_{k+1}. Hence,

$$\tau_{k+1} = \left[\tau_k + \frac{\varphi(\tau_k) - \tau_k}{1 - \varphi'(\tau_k)}\right]_+, \tag{20}$$

where we use the notation $[\alpha]_+ = \max\{0, \alpha\}$.

Lemma 5. *Procedure* (20) *converges to* τ^* *for any initial* $\tau_0 \geq 0$ *in a finite* (*not more than* n) *number of iterations*.

Finally, we can formulate the main result.

Theorem 1. *The set ConvX has* 2^n *vertices. Each vertex* x *can be found by solving scalar equation* (17) *for a given vector* $s \in S$ *via algorithm* (20), *then* $x = A^{-1}y(\tau^*) + x^*$, *where* $y(\tau) = \tau s$ *and* τ^* *is the solution of* (17).

4.2. Algorithm

The algorithm for obtaining k-th vertex of convex hull for polytope X reads as follows.

- Choose the vector $s^k \in S$ (with elements ± 1) that corresponds to k-th orthant of $\mathbb{R}^n$.
- For given s^k find the unique solution τ^* of equation $\tau = \varphi(\tau)$ over $(0, \infty)$ using the iterative method (20).
- Calculate $y = \tau^* s^k$.
- Then $x^k = A^{-1}(y + b)$ is k-th vertex of *ConvX*.

To compute all vertices of *ConvX* one need to apply this algorithm 2^n times. With this vertices we can finally find optimal lower and upper component-wise bounds for solution set X. Indeed, if $\hat{X}^* = \{x^k,\quad k = 1, \dots, 2^n\}$ is the set of all vertices of *ConvX*, then for each component of x^k-s we seek its minimal and maximal value

$$\underline{x}_i = \min_k\{x_i^k\}, \qquad \bar{x}_i = \max_k\{x_i^k\}, \quad i = 1, \dots, n, \tag{21}$$

that gives the optimal interval solution $\boldsymbol{X}^*$ of system (3).

Example 2. For

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix},$$

and $\varepsilon = \delta = 0.25$ the solution set X is a bounded and non-convex polytope illustrated on Fig. 2a. Its linear image after transformation $y = Ax - b$ is shown in Fig. 2b. All vertices of convex hull *ConvY* of the solution set in variables y are represented by vector s^k with elements $s_i^k = \pm 1$ and value τ from (17): $y^1 = (2/3, 2/3)^{\mathrm{T}}$, $y^2 = (-1, 1)^{\mathrm{T}}$, $y^3 = (1, -1)^{\mathrm{T}}$ and $y^4 = (-0.4, -0.4)^{\mathrm{T}}$. By inverse transformation $x = A^{-1}(y + b)$ the vertices of *ConvX* are obtained. And then it is trivial to find interval bounds on X using (21). Finally $\boldsymbol{X}^* = ([-1.5, 2.5], [-0.5, 1.5])^{\mathrm{T}}$.

4.3. Structured uncertainty

In addition to interval description there exist other types of system uncertainty. Structured matrix perturbations bounded in various norms were considered, for example, by Polyak [12]. More general type is linear fractional representation for uncertainty that has been examined by El Ghaoui [3], where the author reduced the search of the optimal outer bounds to semidefinite programming technique.

A family of interval linear systems with matrix uncertainty in a structured form

$$(A + B\Delta_A C)x = b + B\Delta_b \tag{22}$$

is also tractable with our method. Here, Δ_A is a rectangular matrix bounded in ∞-norm $\|\Delta_A\|_\infty \le \varepsilon$ and $\|\Delta_b\|_\infty \le \delta$. Then the characterization for the solution set is rewritten as

$$X = x^* + A^{-1}Y, \qquad Y = \{y : \|y\|_\infty \le \varepsilon\|Cx^* + CA^{-1}By\|_1 + \delta\}. \tag{23}$$

All the results obtained remain valid for this case. Theorem 1 and the algorithm hold true if matrix A^{-1} and vector $x^* = A^{-1}b$ are replaced by $CA^{-1}B$ and $CA^{-1}b$, respectively. Choosing in (22) $B = \mathrm{diag}\{\beta_1, \dots, \beta_n\}$, $C = \mathrm{diag}\{c_1, \dots, c_n\}$ we get $(B\Delta_A C)_{ij} = \beta_i c_j (\Delta_A)_{ij}$. Thus entries w_{ij} of perturbation matrix $W = B\Delta_A C$ are subject to weighted interval constraints:

$$|w_{ij}| \le \beta_i c_j \varepsilon, \tag{24}$$

which are called in [13] rank-one perturbations. We conclude that model (22) covers rank-one interval perturbations. However, it does not cover the general case of arbitrary weighted interval constraints

$$|w_{ij}| \le \alpha_{ij}\varepsilon, \quad \alpha_{ij} \ge 0, \tag{25}$$

(a)

(b)

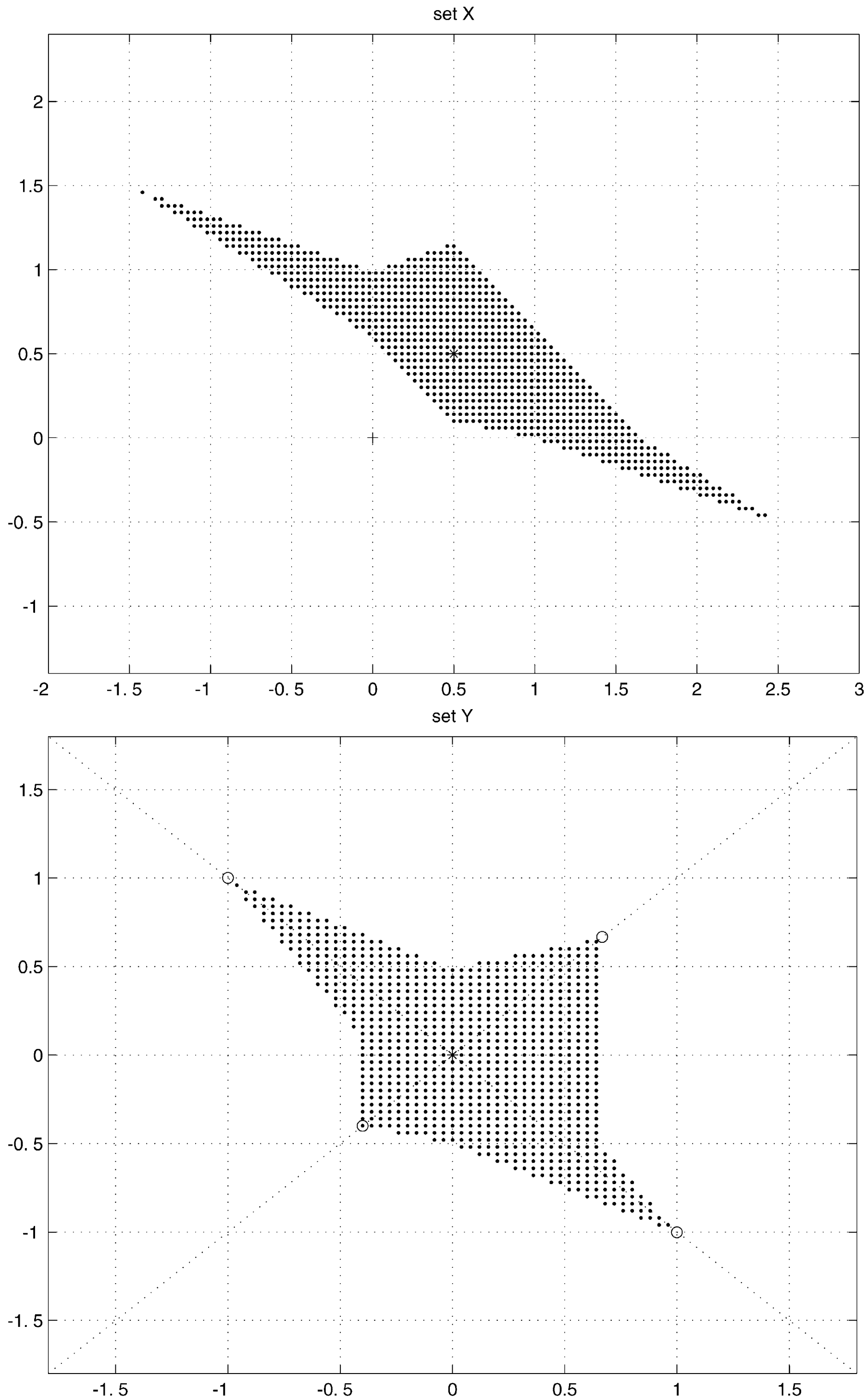

Fig. 2. (a) Solution set X. (b) Solution set Y.

which are considered in [13]. Notice also that the interval system with so-called linear fractional representation of uncertainty (see e.g. [3])

$$(A + B\Delta(I - H\Delta)^{-1}C)x = b \tag{26}$$

can be treated in the same way; the matrix A^{-1} in the algorithm should be replaced by $H + CA^{-1}B$.

5. Interval overbounding

As we have seen above, the calculation of the optimal interval solution may be hard for large-dimensional problems. Hence, its simple interval overbounding is of interest. We provide below two such estimates.

According to the inequality $\|y\|_\infty \le \varepsilon\|A^{-1}(y + b)\|_1 + \delta$ for the set Y we write

$$\|A^{-1}(y + b)\|_1 \le \|Gy\|_1 + \|x^*\|_1 \le \|G\|_{\infty,1}\,\|y\|_\infty + \|x^*\|_1, \tag{27}$$

where $G = A^{-1}$. Therefore

$$\|y\|_\infty \le \gamma = \frac{\varepsilon\|x^*\|_1 + \delta}{1 - \varepsilon\|A^{-1}\|_{\infty,1}}. \tag{28}$$

It means that all $y \in Y$ lie inside a ball in ∞-norm of radius γ. This ball is the first overbounding interval estimate. In most cases (28) is the minimal ball centered at the origin containing Y.

The main difficulty here is to calculate $(\infty, 1)$ matrix norm; this is again NP-hard problem. There exist tractable upper bounds for this norm [1,8]; we use the simplest one: for a given matrix G the value $\|G\|_{\infty,1}$ can always be approximated by 1-norm:

$$\|G\|_{\infty,1} = \max_{\|x\|_\infty \le 1} \|Gx\|_1 = \max_{\|x\|_\infty \le 1} \sum_{i=1}^{n} \Big|\sum_{j=1}^{n} g_{ij}x_j\Big| \le \sum_{i,j=1}^{n} |g_{ij}| = \|G\|_1. \tag{29}$$

Hence, the inequality (28) is rewritten as

$$\|y\|_\infty \le \frac{\varepsilon\|x^*\|_1 + \delta}{1 - \varepsilon\|A^{-1}\|_1}, \tag{30}$$

where ε should be already less than $1/\|A^{-1}\|_1$. An interval estimate for Y implies an interval estimate for X. Indeed, x is a linear function of $y : x = x^* + A^{-1}y$ and optimization of x_k (5) on a cube can be performed explicitly and we arrive to the following result.

Theorem 2. *The interval vector* $\boldsymbol{X} = ([\underline{x}_1, \bar{x}_1], \ldots, [\underline{x}_n, \bar{x}_n])^{\mathrm{T}}$ *with*

$$\underline{x}_k = x_k^* - \gamma\|g_k\|_1, \quad \bar{x}_k = x_k^* + \gamma\|g_k\|_1 \tag{31}$$

contains the solution set X, *where* g_k *is the k-th row of* $G = A^{-1}$ *while* γ *is the right-hand side of* (28) *or* (30).

Thus calculation of $\boldsymbol{X} \supseteq \boldsymbol{X}^*$ given by (31), (30) is not involved, it yields neither combinatorial difficulties nor requires solution of linear programming problems. Numerous examples confirm that this overbounding solution is close to optimal.

Example 3. Consider the linear system $Hx = b$ with Hilbert $n \times n$-matrix. Recall that H is called Hilbert matrix of order n if its entries $h_{ij} = 1/(i + j - 1),\ i, j = 1, \dots, n$:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

This matrix is poorly conditioned even for small dimensions. For this reason, serious difficulties arise in computation of the solutions of linear equations with this matrix because of the big sensitivity of the solution to errors in the data [4]. Thus the Hilbert matrix is a good test example in the framework of interval uncertainty.

Let $b = (1, 1/2, 1/3, \dots)^{\mathrm{T}}$, then $x^* = (1, 0, 0, \dots)^{\mathrm{T}}$. The inverse Hilbert matrix T has large integer elements; it can be calculated via the procedure *invhilb* in *MATLAB*. For instance, for $n = 5$ T reads

$$T = H^{-1} = \begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix}. \tag{33}$$

Let us take $\varepsilon = \delta = 10^{-7}$ (that is data are correct up to seven decimal digits). Then direct application of algorithm of section 4.2 (which requires 32 solution of one-dimensional equations; solution of each equation was obtained in a single iteration for $\tau_0 = 0$) provides

$$\boldsymbol{X}^* = \begin{pmatrix} [0.99924758, 1.00075299] \\ [-0.01403808, 0.01402751] \\ [-0.06046547, 0.06051100] \\ [-0.09139344, 0.09132468] \\ [-0.04468784, 0.04472149] \end{pmatrix}, \tag{34}$$

while estimates (31), (30) and (31), (28) coincide ($\|T\|_{\infty,1} = \|T\|_1$) and give

$$\boldsymbol{X} = \begin{pmatrix} [0.99924701, 1.00075299] \\ [-0.01403808, 0.01403808] \\ [-0.06051100, 0.06051100] \\ [-0.09139344, 0.09139344] \\ [-0.04472149, 0.04472149] \end{pmatrix}. \tag{35}$$

We conclude that in this case the simplest overbounding (31), (30) is a very precise approximation of the smallest interval solution. The same remains true for larger dimensions. For instance, if $n = 10,\ \ \varepsilon =$

$\delta = 10^{-14}$ we have $\|T\|_1 \approx 4 \cdot 10^{13}$ and estimate (31), (30) provides a good approximation to the optimal interval solution

$$(1 + [-10^{-6}, 10^{-6}], \dots , [-0.402, 0.402], [-0.211, 0.211], [-0.046, 0.046])^{\mathrm{T}}.$$

Note that for very high accuracy in data ($\varepsilon = \delta = 10^{-14}$) the accuracy of the solution of the interval equation is inaccessible; this is numerical confirmation of ill-posedness of equations with Hilbert matrix even for moderate dimensions.

6. Summary

In this paper we considered the solution set and obtained the optimal interval solution for interval system of linear algebraic equations as well as its overbounding estimates. The search of optimal interval solution is reduced to checking 2^n vertices for convex hull of the solution set. Each vertex can be found by solving one scalar equation. To solve large-scaled interval systems one can apply overbounding technique.

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References

[1] A. Ben-Tal, A. Nemirovski, On approximating matrix norms, Math. Progr., in press.
[2] J.E. Cope, B.W. Rust, Bounds on solutions of linear systems with inaccurate data, SIAM J. Numer. Anal. 16 (1979) 950–963.
[3] L. El Ghaoui, Inversion error, condition number, and approximate inverses of uncertain matrices, Linear Algebra Appl. 343-344 (2002) 171–193.
[4] G.E. Forsythe, C.B. Moler, Computer Solution of Linear Algebraic Systems, Prentice-Hall, Englewood Cliffs, NJ, 1967.
[5] G.H. Golub, C.F. Van Loan, Matrix Computations, Johns Hopkins University Press, Baltimore, 1996.
[6] N.J. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Philadelphia, 1996.
[7] V. Kreinovich, A.V. Lakeev, S.I. Noskov, Optimal solution of interval linear systems is intractable (NP-hard), Interval Comput. 1 (1993) 6–14.
[8] Yu. Nesterov, Semidefinite relaxation and nonconvex quadratic optimization, Optimization Software 9 (1998) 141–160.
[9] A. Neumaier, Interval Methods for Systems of Equations, Cambridge University Press, Cambridge, 1990.
[10] W. Oettli, W. Prager, Compatibility of approximate solution of linear equations with given error bounds for coefficients and right-hand sides, Numer. Math. 6 (1964) 405–409.
[11] W. Oettli, On the solution set of a linear system with inaccurate coefficients, SIAM J. Numer. Anal. 2 (1965) 115–118.
[12] B.T. Polyak, Robust linear algebra and robust aperiodicity, in: A. Rantzer, C. Byrnes (Eds.), Directions in Mathematical System Theory and Optimization, Springer, Berlin, 2002, pp. 249–260.
[13] J. Rohn, Systems of linear interval equations, Linear Algebra Appl. 126 (1989) 39–78.
[14] J. Rohn, Overestimations in bounding solutions of perturbed linear equations, Linear Algebra Appl. 262 (1997) 55–65.
[15] B.W. Rust, W.R. Burrus, Mathematical Programming and the Numerical Solution of Linear Equations, Elsevier, New York, 1972.
[16] S.P. Shary, On optimal solution of interval linear equations, SIAM J. Numer. Anal. 32 (1995) 610–630.