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Rejection of Bounded Disturbances via Invariant Ellipsoids Technique

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Abstract— **In this paper an approach based on invariant ellipsoids is applied to the problem of persistent disturbance rejection by means of static state-feedback control. Dynamic system is supposed to be linear time-invariant and affected by unknown-but-bounded exogenous disturbances. Synthesis of an optimal controller that returns a minimum of the size of the corresponding invariant ellipsoid is reduced to one-dimensional convex minimization with LMI constraints. The problem is considered in continuous and discrete time cases.**

I. INTRODUCTION

Unknown-but-bounded description of uncertain variables forms the guaranteed set-membership approach to the problems appeared in system and control theory. This approach has received much attention as the main alternative to stochastic techniques that have been developed for estimation, control and identification. The key point of it is that there is no need to know the statistical distribution of model errors and disturbances except only its lower and upper bounds or set values. This assumption looks to be more acceptable in practice in many situations.

This approach was initially established by Bertsekas and Schweppe, see [2], [3] and [13] as basic references. Its concept is based on the analysis of reachable and feasible sets for uncertain dynamic models or on the search of their approximations by simple convex domains like boxes, polyhedra, ellipsoids, that has generated a variety of further works on this and related topics. An attractive tool in the set-membership framework is the notion of invariance and invariant (or positively invariant) sets. A set in the state space is said to be *positively invariant* for a given dynamic system if every trajectory initiated in this set remains inside it at all future time instants. Positively invariant set has the property that it always contains the reachable set of dynamic system. In spite of conservatism in approximations of reachable domains, invariant sets have a close relation to Lyapunov functions, and for this reason they are quite useful for the synthesis of feedback control. A good reference in this sense is the survey paper [5], which gives a broad outlook of applications of invariant set theory in automatic control. Among the different special families of positively invariant sets, a particular class of ellipsoidal invariant sets can be emphasized. The main advantage of invariant ellipsoids lies in its simple characterization as a solution of parametrized linear matrix inequalities. Therefore the optimal control problems in this description can be reduced to the semidefinite programs, i.e. to the optimization of a linear function under LMI constraints. This optimization problem is convex and is now a powerful tool in many control applications [6].

Using the technique of invariant ellipsoids in this paper the problem of persistent disturbance rejection is considered. The question of how to compensate the effect of persistent unknown-but-bounded disturbances by means of feedback control is very important in system engineering. First type of methods appeared on this topic is founded on dynamic programming technique [10], [2], [9]. Another one is the popular l_1-optimal control theory [16], [8], which is formulated in terms of the worst case peak-to-peak gain minimization. Alternative to the dynamic programming and l_1 approaches, which often suffer of a high complexity of an optimal solution, is the methods based on upper bounds of l_1–norm such as the so-called $*$-norm introduced in [1]. Minimization of the $*$-norm allows the determination of a fixed-order controllers that compensate the disturbance influence. Some properties of this norm are discussed in [15], in particular it is shown that it can be very conservative upper bound for l_1-norm. Direct analogues of the $*$-norm are invariant sets for dynamic system. The approach established invariant sets for the disturbance rejection has already received some attention [4], [14]. We believe that invariant ellipsoids technique is a challenging alternative to l_1 approach, because the last one is based on the assumption $x(0) = 0$, and nonzero initial conditions can cause serious troubles. For instance, the examples in [15] demonstrate non-robustness with respect to $x(0) \neq 0$. In contrast, invariant ellipsoids automatically cover nonzero initial conditions.

The objective of the present paper is to use the LMI technique to treat invariant ellipsoidal sets with its subsequent use for the problem of persistent disturbance rejection. The detailed comparison with the results of [1] is given in Section III.D. The key point of the method is that for linear time-invariant dynamic system the search for the optimal static state-feedback controller can be reduced to semidefinite programming and to one-dimensional convex optimization for continuous and discrete time cases.

II. INVARIANT ELLIPSOIDS: ANALYSIS

Consider an LTI dynamic system

$$\begin{aligned} \dot{x} &= Ax + Bw, \qquad (1)\\ y &= Cx \end{aligned}$$

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with $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^l$. It is assumed that matrix A is stable (its eigenvalues have negative real part) and $\|w(t)\| \leq 1$, $\forall t \geq 0$, where $\|\cdot\|$ is the Euclidean vector norm. Denote the non-degenerate ellipsoid in $\mathbb{R}^n$ centered at the origin as

$$\mathscr{E} = \left\{x \in \mathbb{R}^n : \, x^T P^{-1} x \leq 1\right\}, \quad P > 0. \tag{2}$$

Definition 1: Ellipsoid $\mathscr{E}$ is said to be positively invariant for dynamic system (1) if $x(0) \in \mathscr{E}$ implies $x(t) \in \mathscr{E}$, $\forall t > 0$ for every system trajectory x. □

In this paper we put our attention only on the case of non-degenerate invariant ellipsoids (with non-empty interior). For this reason, the condition of controllability for matrix pair (A,B) is supposed to be satisfied for the system (1), i.e. $\mathrm{rank}\left[B\, AB \ldots A^{n-1}B\right] = n$.

It is obvious that for the dynamic system under consideration the invariant ellipsoid $\mathscr{E}$ always contains the reachable set

$$\mathscr{R} = \{x = x(t) : \, \dot{x} = Ax + Bw, \, x(0) = 0, \, \|w(t)\| \leq 1, \, t \geq 0\}.$$

This notion is used in [7] as a basic tool for construction of the ellipsoidal state estimation methods for continuous-time models. A family of invariant ellipsoids is defined by the following theorem.

Theorem 1 ([1], [6]): The non-degenerate ellipsoid $\mathscr{E}$ is positively invariant for the stable dynamic system (1) with $\|w(t)\| \leq 1, \forall t \geq 0$ if and only if $P > 0$ satisfies

$$AP + PA^T + \alpha P + \alpha^{-1} BB^T \leq 0 \tag{3}$$

for some $\alpha > 0$ where (A,B) is controllable. □

We are interested in the minimal invariant ellipsoids. If dynamic system is stable, then there exists a unique invariant ellipsoid, which minimizes some certain criterion $\varphi(P)$. The most natural measures of size for ellipsoids are: $f_1(P) = \mathrm{Tr}\, P$, $f_2(P) = \mathrm{Det}\, P$, or $f_3(P) = \|P\|$ — spectral matrix norm. As a criterion in this paper we take $\varphi(P) = \mathrm{Tr}\, CPC^T$. Minimization of this objective function corresponds to the search for the minimal trace invariant ellipsoid for the system output $y(t) = Cx(t)$.

Theorem 2 ([1]): All minimal invariant ellipsoids of stable dynamic system (1) with controllable pair (A,B) belong to the one-parameter family of ellipsoids with matrices $P(\alpha)$ for $0 < \alpha < \alpha^*$, where $P(\alpha) > 0$ is the solution of the Lyapunov equation

$$AP + PA^T + \alpha P + \alpha^{-1} BB^T = 0 \tag{4}$$

and $\alpha^* = -2 \max(\mathrm{Re}\, \lambda_i(A))$, $\lambda_i(A)$ are eigenvalues of matrix A. Moreover, the function $f(\alpha) = \mathrm{Tr}\, CP(\alpha)C^T$ is strictly convex on the interval $0 < \alpha < \alpha^*$. □

Therefore the one-dimensional minimization

$$\begin{array}{ll} & \min\limits_{0<\alpha<\alpha^*} \; \mathrm{Tr}\, CP(\alpha)C^T \\ \text{subj. to} & AP + PA^T + \alpha P + \alpha^{-1} BB^T = 0 \end{array} \tag{5}$$

is strictly convex over $0 < \alpha < \alpha^*$ and has a unique solution on this interval. On the other hand, this solution can alternatively be obtained by solving the scalar equation $f'(\alpha) = 0$ with $f'(\alpha) = \mathrm{Tr}\, CG(\alpha)C^T$ and $G(\alpha)$ from

$$AG + GA^T + \alpha G + P - \alpha^{-2} BB^T = 0.$$

The majority of results on the analysis of the invariant sets and its application in control deals with the continuous-time dynamic systems. However the above statements have the discrete-time counterparts. For LTI discrete-time dynamic system

$$\begin{aligned} x_{k+1} &= Ax_k + Bw_k, \\ y_k &= Cx_k \end{aligned} \tag{6}$$

assume that the matrix A is stable (its eigenvalues lie inside the unit disc) and $\|w_k\| \leq 1$ for all $k = 0, 1, 2, \ldots$ Denote by $\rho(A) = \max|\lambda_i(A)|$ the spectral radius of matrix A. Also suppose that the matrix pair (A,B) is controllable.

Theorem 3: Let $\dim(x) \geq 2$, matrix $A \in \mathbb{R}^{n\times n}$ be invertible, $\rho(A) < 1$, $B^T B > 0$ and the pair (A,B) is controllable. The ellipsoid $\mathscr{E}$ defined in (2) is positively invariant for the dynamic system (6) with $\|w_k\| \leq 1$, $\forall k \geq 0$, if and only if

$$\frac{1}{\alpha} APA^T - P + \frac{1}{1-\alpha} BB^T \leq 0 \tag{7}$$

with some $\alpha \in (\rho(A)^2, 1)$. □

The proof is based on the version of S-theorem with two quadratic constraints presented in [11].

Proof: Evidently, x belongs to the positively invariant ellipsoid $\mathscr{E}$ (2) iff

$$(Ax + Bw)^T P^{-1} (Ax + Bw) \leq 1 \tag{8}$$

$$\forall\, (x,w) : \quad x^T P^{-1} x \leq 1, \quad w^T w \leq 1. \tag{9}$$

Denote $Q = P^{-1}$ and apply Theorem 4.1 from [11], which says that conditions (8)–(9) are equivalent to the existence of such $\tau_1, \tau_2 \geq 0$, $\tau_1 + \tau_2 \leq 1$ that

$$\begin{bmatrix} \tau_1 Q - A^T QA & -A^T QA \\ -B^T QB & \tau_2 I - B^T QB \end{bmatrix} \geq 0 \tag{10}$$

Notice that this LMI implies $\tau_1, \tau_2 > 0$, since $Q > 0$, $A^T QA > 0$, and $B^T QB > 0$. Moreover, $\tau_2 \geq \lambda_{\max}(B^T QB) > 0$. It suffices to consider $\tau_2 > \lambda_{\max}(B^T QB)$ (in case equality here one may add an arbitrary small $\varepsilon > 0$ to τ_2 in (10), repeat all the algebra below and finally tend ε to zero). Then LMI (10) is equivalent to

$$\tau_1 Q - A^T QA \geq A^T QB(\tau_2 I - B^T QB)^{-1} B^T QA \tag{11}$$

By the Matrix Inverse Lemma,

$$(Q^{-1} - \tau_2^{-1} BB^T)^{-1} = Q + QB(\tau_2 I - B^T QB)^{-1} B^T Q,$$

and (11) may be written as $\tau_1 Q \geq A^T (Q^{-1} - \tau_2^{-1} BB^T)^{-1} A$, or, equivalently,

$$P \geq \tau_1^{-1} APA^T + \tau_2^{-1} BB^T. \tag{12}$$

On the contrary, LMI (12) implies

$$I = Q^{1/2} P Q^{1/2} > \tau_2^{-1} Q^{1/2} BB^T Q^{1/2}$$

and $\tau_2 > \lambda_{\max}(Q^{1/2} BB^T Q^{1/2}) = \lambda_{\max}(B^T QB)$. Thus, LMIs (10) and (12) are equivalent.

Now, one may easily see that it is suffice to ensure the LMI (12) only for maximal $\tau_2 = 1 - \tau_1$, for a fixed $\tau_1 \in (0,1)$. Putting $\alpha = \tau_1$ reduces (12) to (7). If pair (A,B) is controllable with a stable matrix A then each solution P of the LMI (7) is positive-definite. □

Similar to Theorem 2 the search for minimal invariant ellipsoid with respect to some scalar criterion can be reduced to the search of the minimal ellipsoid in the one-parameter family $P(\alpha)$ defined by the discrete-time Lyapunov equation

$$\frac{1}{\alpha}APA^T - P + \frac{1}{1-\alpha}BB^T = 0. \tag{13}$$

Theorem 4: Let (A,B) be controllable and $P(\alpha) > 0$ be a solution of Lyapunov equation (13) on the interval $\rho(A)^2 < \alpha < 1$. Then function $f(\alpha) = \mathrm{Tr}\, CP(\alpha)C^T$ is strictly convex on this interval. □

Hence, the one-dimensional minimization

$$\begin{array}{rl} \min\limits_{\rho(A)^2<\alpha<1} & \mathrm{Tr}\, CP(\alpha)C^T \\ \text{subj. to} & \dfrac{APA^T}{\alpha} - P + \dfrac{BB^T}{1-\alpha} = 0 \end{array} \tag{14}$$

is strictly convex over $\rho(A)^2 < \alpha < 1$ and provides the minimal trace invariant ellipsoid for system output $y_k = Cx_k$. This minimal ellipsoid can be also obtained in another way as a solution of scalar equation $f'(\alpha) = 0$ with $f'(\alpha) = \mathrm{Tr}\, CG(\alpha)C^T$ and $G(\alpha)$ from

$$\frac{1}{\alpha}AGA^T - G - \frac{1}{\alpha^2}APA^T + \frac{1}{(1-\alpha)^2}BB^T = 0.$$

Example 1: Take the discrete-time system (6) with

$$A = 0.95\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}, \quad \beta = \pi/15,$$

and $B = C = I$, where I is the identity (2×2)-matrix. Since matrix A is stable (its spectral radius $\rho(A) = 0.95 < 1$) and matrix pair (A,B) is controllable, then the Lyapunov equation (13) has a unique positive definite solution for every fixed α such that $\rho(A)^2 < \alpha < 1$. These solutions form the one-parameter family of invariant ellipsoids of the system. The minimal invariant ellipsoid $\mathscr{E}_{\min}$ of the system belongs to this family and is obtained via a scalar convex optimization. Here $\mathscr{E}_{\min}$ represents a ball of radius $r = 20$ (bold-faced line on Fig. 1). As an example, the system trajectory initiated at $x_0 = (-1,0)^T$ and corrupted by disturbance vector $w_k = Ax_k/\|Ax_k\|$ is represented on the figure. The disturbance w_k is the unite vector that is always codirectional with Ax_k for all $k = 0,1,2,\ldots$ Therefore this trajectory tends to the boundary of $\mathscr{E}_{\min}$ as $k \to \infty$ but it never exceeds the boundary of this minimal invariant ellipsoid. □

III. Invariant ellipsoids: synthesis

In order to compensate the influence of persistent unknown-but-bounded disturbances on model output in LTI dynamic system, a static state-feedback control is introduced in this section. Minimization of this influence leads to the search of the minimal invariant ellipsoid for the closed-loop system as described previously. The continuous-time case and discrete-time case are considered consequently.

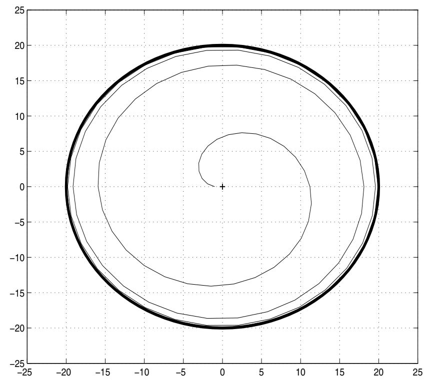

Fig. 1. The minimal invariant ellipsoid.

A. Continuous-time case

Let us consider a dynamic system described by

$$\begin{array}{rcl} \dot{x} & = & Ax + B_1 u + Dw, \\ y & = & Cx + B_2 u, \\ u & = & Kx. \end{array} \tag{15}$$

Matrix A is not assumed to be stable, but the pair (A,B_1) is supposed to be controllable, and $\|w(t)\| \le 1$. Also assume $B_2^T C = 0$. The goal is to find a static controller K, which stabilize the closed-loop system and optimally reject the influence of unknown-but-bounded disturbances $w(t) \in \mathbb{R}^m$ in the sense of minimizing the size of the invariant ellipsoid $\mathscr{E}_y = \{y : y^T(CPC^T)^{-1}y \le 1\}$ for the output vector $y(t) \in \mathbb{R}^l$.

Note here that with $B_2 = 0$ this problem can have no solution that is illustrated by the following example.

Example 2: Let $n = m = l = 1$ and take

$$\begin{array}{rcl} \dot{x} & = & ax + u + w, \\ y & = & x, \end{array}$$

where $|w(t)| \le 1$. Choose $u = kx$. Then $\dot{x} = (a+k)x + w$ and

$$x(t) = x(0)e^{-\alpha t} + \int_0^t e^{-\alpha\tau} w(t-\tau)d\tau,$$

here $\alpha = -(a+k) > 0$. Thus, if for an arbitrary $\varepsilon > 0$ the initial $|x(0)| \le \varepsilon$, then

$$\begin{array}{rl} |x(t)| & \le |x(0)|e^{-\alpha t} + \displaystyle\int_0^t e^{-\alpha\tau} d\tau \\ & \le \left(\varepsilon - \dfrac{1}{\alpha}\right) e^{-\alpha t} + \dfrac{1}{\alpha} \le \max\left\{\varepsilon, \dfrac{1}{\alpha}\right\}. \end{array}$$

From this expression it follows that for any gain k such that $k < -a$, i.e. $\alpha = -(a+k) > 0$, all the invariant intervals in the state space are defined by $|x(t)| \le \varepsilon$ under $\varepsilon \ge -(a+k)^{-1}$. Then the size of the minimal invariant interval $\varepsilon_{\min} = -(a+k)^{-1}$ tends to zero as $k \to -\infty$. Therefore, there exist no optimal finite k, and the problem of the optimal persistent disturbance rejection by static state-feedback has no solution in this case.

On the other hand, take the system output affected directly by the control

$$y = x + bu = (1+bk)x, \quad b \ne 0,$$

where $u = kx$. This leads to the inequality

$$|y(t)| \leq |1+bk| \cdot \max\{\varepsilon, 1/\alpha\}.$$

The minimal size invariant interval for system output $y(t)$ is equal to

$$|1+bk|\varepsilon_{\min} = -|1+bk|/(a+k) = r(k).$$

Function $r(k)$ has the only minimum under $k < -a$, namely, $k_{\min} = -1/b$ if and only if $a < 1/b$. In any other case, function $r(k)$ is either constant (when $ab = 1$) or monotone (under $k < -a$) and has an exact lower bound

$$\inf_{k<-a} r(k) = |b|.$$

Thus, the problem of finding k minimizing system output $y(t)$ under $a < 1/b$ admits finite solution. □

Let us further study the MIMO case. The system (15) is rewritten as

$$\begin{aligned} \dot{x} &= (A+B_1K)x + Dw, \\ y &= (C+B_2K)x, \end{aligned} \tag{16}$$

that is in the setup of model (1). Then the problem is reduced to (5) after replacing matrix A by $A+B_1K$ and matrix C by $C+B_2K$. It gives us the next optimization

$$\min \ \mathrm{Tr}\ \left[(C+B_2K)P(C+B_2K)^T\right]$$

under

$$(A+B_1K)P + P(A+B_1K)^T + \alpha P + \frac{DD^T}{\alpha} = 0.$$

The last equality is a bilinear matrix equation with respect to P and K. Introduce a new variable $Y = KP$. Since $B_2^TC = 0$, then the objective function to be minimized is rewritten as $\mathrm{Tr}\ \left[CPC^T + B_2YP^{-1}Y^TB_2^T\right]$ subject to the linear equality constraint

$$AP + PA^T + \alpha P + B_1Y + Y^TB_1^T + \frac{DD^T}{\alpha} = 0.$$

The function $f(P,Y) = \mathrm{Tr}\ [YP^{-1}Y^T]$ is convex in matrix variables $P > 0$ and Y. This reduces the problem to multi-dimensional convex optimization.

But from the other side, we can write it in terms of LMIs. Indeed, consider the matrix

$$H = \begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix}, \tag{17}$$

where Z is an auxiliary matrix variable. Then $H \geq 0$ implies $Z \geq YP^{-1}Y^T \geq 0$. Therefore, minimization of $\mathrm{Tr}\ \left[B_2YP^{-1}Y^TB_2^T\right]$ is equivalent to minimization of $\mathrm{Tr}\, B_2ZB_2^T$, and the following result can now be validated.

Theorem 5: The original problem is equivalent to

$$\min \ \mathrm{Tr}\ \left[CPC^T + B_2ZB_2^T\right] \tag{18}$$

subject to LMI constraints

$$\begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix} \geq 0, \quad P > 0,$$

$$AP + PA^T + \alpha P + B_1Y + Y^TB_1^T + \frac{DD^T}{\alpha} = 0. \quad □$$

Note that this is semi-definite programming (SDP) problem for any fixed $\alpha > 0$, and we can exploit standard LMI Toolbox for its numerical solution. Optimization over α is a convex optimization problem.

The state-feedback controller K obtained from Theorem 5 returns the minimum of invariant ellipsoid size (in the sense of trace criterion) for the output of the closed-loop system. However it optimally reduces the influence of the disturbances only for the case when trajectories are initiated within the smallest invariant ellipsoid itself, that can be quite restrictive in many situations. In order to overcome this difficulty, some prior conditions on invariant ellipsoid size can be considered. For instance, let

$$P \geq P_0, \tag{19}$$

where $P_0 > 0$ represents the matrix of lower-bound ellipsoid containing all system trajectories of interest. Inequality (19) can be directly introduced to LMI problem of Theorem 5.

In addition, the constraint on the control action can also be taken into account. Consider $u \in \mathbb{R}^p$, $\mu > 0$, and assume

$$\|u\| \leq \mu. \tag{20}$$

The following lemma reduces the constraint (20) to the equivalent LMI.

Lemma 1: Let $P > 0$ be the matrix of invariant ellipsoid for the system (15) with $\|w\| \leq 1$ and $u = Kx$. Let $Y = KP$. Then the inequality (20) holds true if and only if matrices P and Y satisfy

$$\begin{pmatrix} P & Y^T \\ Y & \mu^2 I \end{pmatrix} \geq 0, \tag{21}$$

where I is the identity matrix of proper dimension. □

Thus, the original problem with additional constraints on invariant ellipsoid size and on the control action is equivalent to the SDP and one-dimensional convex optimization.

B. Discrete-time case

For LTI discrete-time dynamic system

$$\begin{aligned} x_{k+1} &= Ax_k + B_1u_k + Dw_k, \\ y_k &= Cx_k + B_2u_k, \\ u_k &= Kx_k, \end{aligned} \tag{22}$$

where matrix pair (A, B_1) is assumed to be controllable, $\|w(t)\| \leq 1$ and $B_2^TC = 0$, we look for a static controller K that minimizes the size of the invariant ellipsoid $\mathscr{E}_y = \{y_k : y_k^T(CPC^T)^{-1}y_k \leq 1\}$ for the output vector y_k.

As in the continuous-time case system (22) is rewritten

$$\begin{aligned} x_{k+1} &= (A+B_1K)x_k + Dw_k, \\ y_k &= (C+B_2K)x_k, \end{aligned} \tag{23}$$

that is in the framework of the model (6). Then the problem is reduced to (14) after replacing matrix A by $A+B_1K$ and matrix C by $C+B_2K$. It gives us the next optimization problem

$$\min \ \mathrm{Tr}\ \left[(C+B_2K)P(C+B_2K)^T\right]$$

under the equality

$$\frac{(A+B_1K)P(A+B_1K)^T}{\alpha} - P + \frac{DD^T}{1-\alpha} = 0.$$

Introduce a new variable $Y = KP$. Since $B_2^T C = 0$, then the objective function to be minimized is rewritten as $\mathrm{Tr}\left[CPC^T + B_2YP^{-1}Y^TB_2^T\right]$ with the equality constraint

$$\frac{1}{\alpha}\left(APA^T + B_1YA^T + AY^TB_1^T + B_1YP^{-1}Y^TB_1^T\right) - P + \frac{DD^T}{1-\alpha} = 0.$$

Since the function $f(P,Y) = \mathrm{Tr}\left[YP^{-1}Y^T\right]$ is convex in matrix variables $P > 0$ and Y, then the minimization over $0 < \alpha < 1$ is also convex. Let us further write it in terms of LMIs. Consider the matrix

$$H = \begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix}, \quad (24)$$

where Z is an auxiliary matrix variable. Then $H \geq 0$ implies $Z \geq YP^{-1}Y^T \geq 0$. Therefore, minimization of $\mathrm{Tr}\left[YP^{-1}Y^T\right]$ or $\mathrm{Tr}\left[B_2YP^{-1}Y^TB_2^T\right]$ is equivalent to minimization of $\mathrm{Tr}\, B_2ZB_2^T$, and we arrive to the following proposition.

Theorem 6: The original problem is equivalent to

$$\min\ \mathrm{Tr}\left[CPC^T + B_2ZB_2^T\right] \quad (25)$$

subject to the constraints

$$\begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix} \geq 0, \quad P > 0,$$

$$\frac{1}{\alpha}\left(APA^T + B_1YA^T + AY^TB_1^T + B_1ZB_1^T\right) - P + \frac{DD^T}{1-\alpha} = 0. \qquad \square$$

For a fixed $0 < \alpha < 1$ the problem is SDP, while optimization over α is convex. Notice that with the initial conditions on the invariant ellipsoid size (19) and on the control action (20) all results follow the same lines as in the continuous-time case.

C. Example

Consider an example with oscillation of two unite masses connected by elastic spring and sliding without friction along a fixed horizontal rod (see Fig. 2). The control input $u \in \mathbb{R}$ is

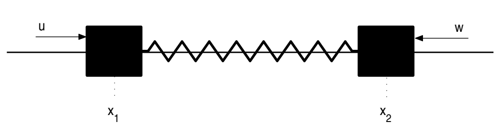

Fig. 2. Double mass-spring system.

applied to the left mass in order to compensate the external disturbance $w \in \mathbb{R}$ affected to the right one. The state vector of the system is $x = (x_1, v_1, x_2, v_2)^T$, where x_1, v_1 and x_2, v_2 are values of coordinate and velocity for the left and right bodies, respectively. The output vector is assumed to be $y = (u, x_2)^T$. Then the corresponding continuous-time model of this double pendulum is described by

$$\begin{array}{rcl} \dot{x}_1 & = & v_1 \\ \dot{v}_1 & = & -x_1 + x_2 + u \\ \dot{x}_2 & = & v_2 \\ \dot{v}_2 & = & x_1 - x_2 - w \\ y & = & (u, x_2)^T \end{array} \iff \begin{array}{rcl} \dot{x} & = & Ax + B_1u + Dw \\ y & = & Cx + B_2u \end{array}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider further the discrete-time approximation of this system

$$\begin{array}{rcl} x_{k+1} & = & \widehat{A}x_k + \widehat{B}_1u_k + \widehat{D}w_k \\ y_k & = & Cx_k + B_2u_k \end{array}$$

with

$$\widehat{A} = e^{\Delta A}, \quad \widehat{B}_1 = \int_0^{\Delta} e^{(\Delta-\theta)A}B_1\,d\theta, \quad \widehat{D} = \int_0^{\Delta} e^{(\Delta-\theta)A}D\,d\theta$$

and with period of time discretization $\Delta = \pi/12$. Matrix pair (A, B_1) is controllable, and $B_2^T C = 0$.

Take $u_k = Kx_k$. Theorem 6 then provides an optimal static state-feedback controller K as a solution of SDP problem and one-parameter minimization; applying LMI Toolbox we get

$$K = \begin{pmatrix} -1.649 & -1.987 & 0.408 & -1.198 \end{pmatrix}.$$

This controller minimizes the output invariant ellipsoid of the closed-loop system due to the trace criterion. Fig. 3 illustrates this optimal invariant ellipsoid $\mathscr{E}_{\min}$ (bolded line) and, as an example, a trajectory of the output oscillations $y_k = (u_k, x_{2,k})^T$ of the pendulum affected by the disturbance $w_k = \sin(k/5)$ and initiated at x_0 that corresponds to a point on the boundary of the output invariant ellipsoid. This trajectory remains inside this invariant ellipsoid. The disturbance behavior w_k and the control law u_k are shown on Fig. 4. This control law is an optimal state-feedback compensator of the influence of the external bounded disturbance w_k on system output y_k obtained by the invariant ellipsoid technique. Fig. 5 and 6 represents another example of the same system but with different disturbance law $w_k = \mathrm{sign}(\sin(k/5))$ and additional constraints on the size of invariant ellipsoids (19) and on the control action (20), where $P_0 = 2I$ and $\mu = 3$.

Notice finally that the overshooting effect often appears when someone looks for a stabilizing controller. This may cause serious troubles. Design of the stabilizing controller that minimizes the invariant ellipsoid of the closed-loop system allows to avoid overshooting.

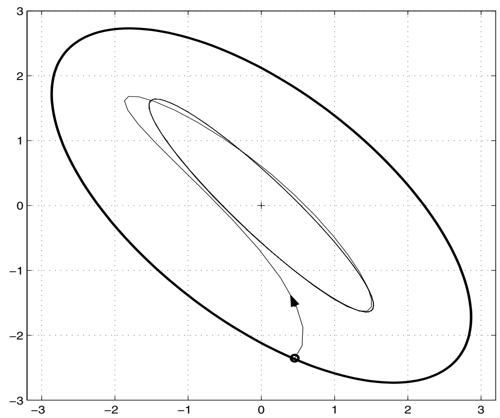

Fig. 3. Trajectory and minimal invariant ellipsoid.

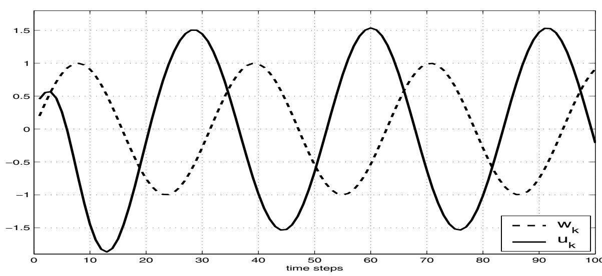

Fig. 4. Disturbance w_k and control u_k.

D. Discussion

It is of interest to compare the above results with the known ones, mainly in [1]. First, we address continuous-time and discrete-time cases simultaneously, while in [1] the authors deal with continuous-time systems only. Second, [1] considers $\|P\|$ as the objective function while we work with $\mathrm{Tr}\, P$, this allows to transform the problems to standard SDP and to simplify the results. Third, the number of parameters in our results is lower (the single parameter α, while there are two parameters in [1]). From technical point of view we rely on more advanced result (S-procedure with two constraints versus standard version of S-procedure with one constraint in [1]).

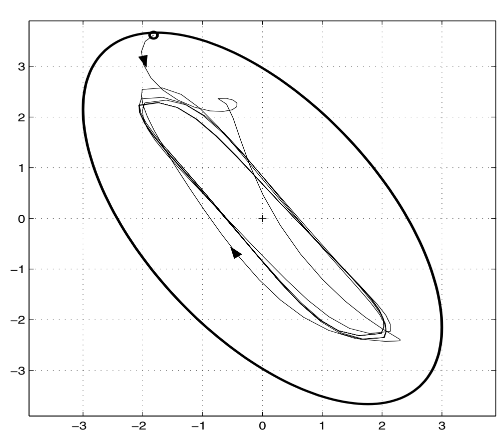

Fig. 5. Trajectory and minimal invariant ellipsoid.

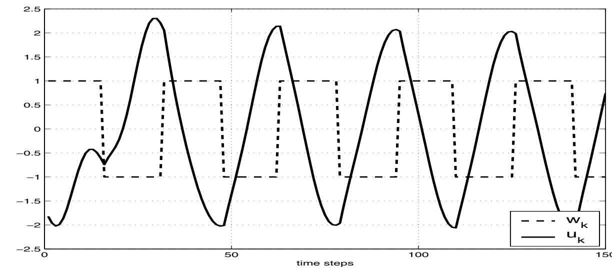

Fig. 6. Disturbance w_k and control u_k.

IV. CONCLUSIONS

The simple method based on invariant ellipsoids technique is proposed for the optimal rejection of unknown-but-bounded disturbances. The static state-feedback controller that returns a minimum of invariant ellipsoid of the closed-loop system is founded via SDP and 1D convex optimization.

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