The geometry and number of the root invariant regions for linear systems

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Abstract

The stability domain is a feasible set for numerous optimization problems. D-decomposition technique is targeted to describe the stability domain in the parameter space for linear parameter-dependent systems. This technique is very simple and efficient for robust stability analysis and design of low-order controllers. However, the geometry of the arising parameter space decomposition into root invariant regions has not been studied in detail; it is an objective of the present paper. We estimate the number of root invariant regions and provide examples, where this number is attained.

Keywords: Control, robustness analysis, characteristic polynomials, stability domain, linear systems.

1 Introduction

Stability is one of the most important properties of control systems. The challenging problem in linear control theory is to describe the total set of parameters (controller coefficients or plant characteristics) providing the stability of the system. The stability domain is a feasible set for numerous optimization problems. The structure of this set can be quite complicated. This leads to approximate solutions of optimization problems. A complete description of the stability domain and its boundary in the parameter space is a step towards an exact solution. Another application is design of low-order controllers, where the order and the parameters of the controller are to be chosen as to ensure the desirable behaviour of the system. Here a complete description of the stability domain is extremely helpful in making a proper choice.

Approaches to these challenging problems can be traced back to the 19th century. In 1876, Vishnegradsky [11] analyzed the construction of the stability domain for third order polynomials with two uncertain parameters. Later, Frazer and Duncan [4] developed the graphical method for the general case of an n^{th}-order polynomial; however, it required finding all roots of the $n \times n$ Hurwitz determinant. The technique in its present form is due to Neimark [7-8] who developed his D-decomposition method for the stability domain analysis. The core of the approach is the decomposition of the parameter space into root invariant regions; the boundaries of the regions are defined by a system of equations. This method is addressed in the books [1], [10]; it is very efficient for low-order controllers design; e.g., see [2].

Until recently, the geometry of D-decomposition has not been well studied. In some particular cases we can be sure that the stability domain is simply connected. However, the structure of the stability domain might be much more complicated. For instance, there are examples with several stability intervals for a gain. The recent paper [9] provides an example, where the stability domain consists of $n-1$ simply connected regions for the case of two uncertain parameters. Several problems arise naturally: how many root invariant regions are there in the parameter space? What is the maximal (minimal) number of the stability regions? In the present paper, we address these problems for characteristic polynomials which depend linearly of one and two uncertain parameters. The boundary of the root invariant regions generated by the D-decomposition method, is an algebraic curve. The topological properties of algebraic curves are the subject of the 16^{th} Hilbert Problem [6], hence it is natural that we exploit certain algebraic geometry tools in our research.

We deal with continuous and discrete-time systems. Throughout the paper we denote by $P(s,\lambda)$ any continuous-time polynomial with an uncertain parameter λ and by $P(z,\lambda)$ any discrete-time polynomial. In the former case, the polynomial is stable iff all its roots have negative real parts, in the latter case, the polynomial is stable iff all its roots are inside the unit circle. Using the mapping $s=\frac{z+1}{z-1}$ we can transform a continuous-time system into a discrete-time one and vice versa.

The paper is organized as follows. In Section 2 we explain the idea of the D-decomposition technique. In Section 3 we analyze polynomial families with one real parameter. A theorem about the maximal number of root invariant regions and the maximal number of stability intervals is stated. The example showing the attainability of this maximal number is given. Sections 4 and 5 are devoted to the cases of one complex parameter and two real parameters, respectively. Several examples demonstrate that the geometry of D-decomposition can be fairly sophisticated. The related results can be found in [5].

2 D-decomposition

Let $P(s,\lambda)$ be a polynomial of degree n with real coefficients $a_k(\lambda)$, where $\lambda\in\mathbb{R}^m$ is an uncertain parameter:

$$P(s,\lambda)=a_n(\lambda)s^n+a_{n-1}(\lambda)s^{n-1}+...+a_0(\lambda).$$

If $P(s,\lambda)$ has k stable and $n-k$ unstable roots, then we say that $\lambda\in D(k)$; i.e. $D(n)$ is a *stability domain*. Simply connected regions of all such domains $D(k)$ generate the decomposition of $\mathbb{R}^m$ into *root invariant regions*, our goal is to describe their boundaries. To abandon $D(k)$ λ should encounter one of the following situations:

1) the polynomial has an imaginary root, that is $P(j\omega, \lambda) = 0$ for some ω,

2) the polynomial has a zero root, i.e. $a_0(\lambda) = 0$,

3) the polynomial changes its degree, i.e. $a_n(\lambda) = 0$.

Thus the boundary of each $D(k)$ can consist of the curves, generated by these three equations (in the first one $\omega \in (-\infty, \infty)$ is considered as a parameter):

$$\begin{array}{l} P(j\omega, \lambda) = 0 \\ a_0(\lambda) = 0, \quad a_n(\lambda) = 0. \end{array} \tag{1}$$

Note that the former equation is equivalent to two real equations (for real and imaginary parts of $P(j\omega, \lambda)$). Equations (1) define *D-decomposition* of the parameter space — they characterize the boundary of root invariant regions $D(k)$. We do not discuss here how to find k for each region (this issue will be addressed later).

The same technique can be used to construct a boundary of the regions with certain number of real roots. Indeed, the number of real roots can change when a multiple root arises, i.e. for some $s \in \mathbb{R}$

$$\begin{array}{l} P(s, \lambda) = 0, \\ P'(s, \lambda) = 0, \end{array} \tag{2}$$

where the prime sign stands for the derivative.

Similar equations can define the domain of *aperiodic stability*, that is the set of parameters which guarantee that all roots are stable and real.

Example (Vyshnegradsky, 1876). We consider a cubic polynomial reduced to the form of Vyshnegradsky (i.e. with $a_3 = 1, \ a_0 = 1$):

$$P(s, \lambda_1, \lambda_2) = s^3 + \lambda_2 s^2 + \lambda_1 s + 1. \tag{3}$$

Solving the system of equations (2) we obtain the parametrized curve

$$\begin{array}{l} \lambda_1(s) = -\frac{2}{s} + s^2, \\ \lambda_2(s) = \frac{1}{s^2} - 2s, \end{array}$$

where s is a real parameter. Equations (1) give a nonparametric formula for the boundary of the stability domain $\lambda_1 \lambda_2 > 1$. Figure 1 depicts the stability domain and the regions having different numbers of real roots in the $\{\lambda_1, \lambda_2\}$-plane (they are marked by digits).

3 One real parameter

Consider the polynomial family with one real parameter

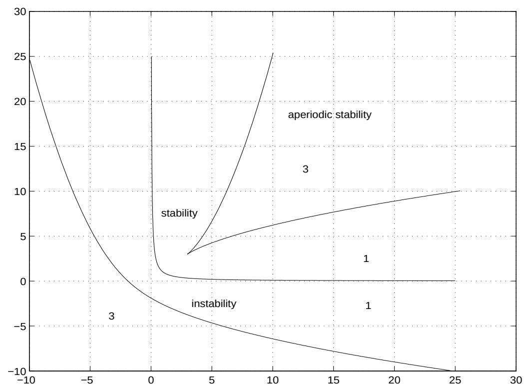

Figure 1: D-decomposition for cubic polynomial

$$P(s,\lambda) = \{a(s) + \lambda b(s),\ \lambda \in \mathbb{R}\}, \tag{4}$$

where $a(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$ and $b(s) = b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0$ are given polynomials of degree n with real coefficients. Since λ varies, the number of stable roots of the polynomial can range over $0, ..., n$. Below we estimate the number of λ that are the root invariant region boundary.

Theorem 1 *The polynomial family (4) can have no more than $n+1$ root invariant regions and no more than $\lceil \frac{n}{2} \rceil$ stability intervals.*

Proof. The boundary of the root invariant regions is given by the equation $P(j\omega, \lambda)$=0, $0 \leq \omega \leq \infty$. We rewrite it in the form:

$$P(j\omega, \lambda) = U_0 + \lambda U_1 + j\omega(V_0 + \lambda V_1) \tag{5}$$

where

$$\begin{array}{c} U_0 = a_0 - a_2\omega^2 + a_4\omega^4 + ... \\ V_0 = a_1 - a_3\omega^2 + a_5\omega^4 + ... \\ U_1 = b_0 - b_2\omega^2 + b_4\omega^4 + ... \\ V_1 = b_1 - b_3\omega^2 + b_5\omega^4 + ... \end{array}$$

A solution of two linear equations with one variable $\lambda = \lambda(\omega)$ exists if and only if

$$U_0 V_1 - U_1 V_0 = 0. \tag{6}$$

The left-hand side of (6) is an $n-1$ order polynomial in ω^2. So there exist no more than $n-1$ different real values of $\lambda = -\dfrac{U_0}{U_1} = -\dfrac{V_0}{V_1}$ that define the boundary of the root invariant regions.

According to (1), there can be two extra boundary points of the root invariant regions. One such value $\lambda = -a_n/b_n$ appears as the degree of the polynomial changes; another value $\lambda = -a_0/b_0$ corresponds to $\omega = 0$.

These $n+1$ points divide the λ axis into $n+1$ root invariant intervals (the intervals $\lambda \longrightarrow +\infty$ and $\lambda \longrightarrow +\infty$ are regarded as the same interval). Since any two neighboring intervals can not be both stability intervals, we conclude that there can be no more than $\lceil \frac{n}{2} \rceil$ stability intervals. $\diamond$

We suggest a purely algebraic (not graphical) algorithm to calculate the number of stable roots in every root invariant region.

Algorithm 1

i. Order the solutions of equation (6) and the two extra values $\lambda = -a_n/b_n$, $\lambda = -a_0/b_0$ as follows: $\lambda_1 < \lambda_2 < ... < \lambda_s$.

ii. When $\lambda < \lambda_1$, $P(s, \lambda)$ has the same number of stable roots as $b(s)$ (this number is easy to find).

iii. We proceed by increasing λ. When λ crosses one of the λ_i values, one or two more roots become stable or unstable. For any of the two extra values only one root moves. For all other ω_i two roots cross the imaginary axis. If $\left.\frac{d\ Im(\lambda(\omega))}{d\omega}\right|_{\omega^*} < 0$, the roots become stable, and unstable if the derivative is positive. So adding and subtracting the proper number of roots, we find the number of stable roots in every root invariant region.

The result is also valid for discrete-time case. Let us consider an example with the maximal number of root invariant regions.

Example 1 The uncertain polynomial

$$P(z, \lambda) = z^n + \lambda z^{n-1} + \varepsilon z^{n-2} + \alpha, \tag{7}$$

where $1 < \varepsilon < 1 + \frac{2}{(n-2)^2}$, $\alpha = 1 - \varepsilon - \frac{1}{n^2}$, has $\lfloor \frac{n}{2} \rfloor$ stability intervals (Fig. 2).

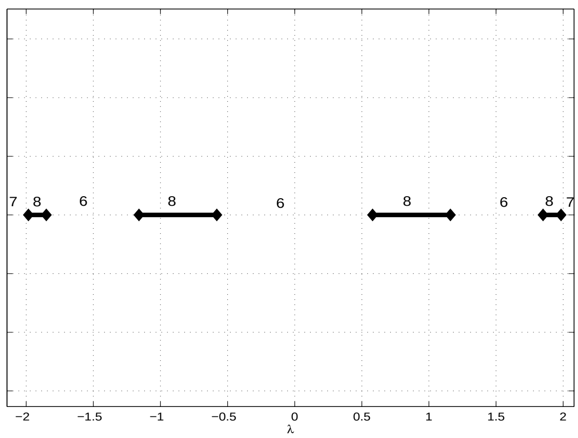

Figure 2: $\frac{n}{2}$ stability intervals in Example 1

The root boundary of the invariant regions is defined by

$$\lambda = -e^{j\omega} - \varepsilon e^{-j\omega} - \alpha e^{-(n-1)j\omega},\ Im(\lambda) = 0.$$

The latter equation

$$(\varepsilon - 1)\sin\omega + \alpha\sin(n-1)\omega = 0$$

has n solutions in the segment $[0, \pi]$ because $|\alpha| > |\varepsilon - 1|$. When $\lambda < \lambda_1^*$, the polynomial has one unstable root and the sign of the derivative alternates at boundary points. As a result, we can obtain the maximal number of the stability intervals in this example.

It is interesting to find out, what is the minimal number of the root invariant regions. The answer is trivial.

Example 2 We consider

$$P(s, \lambda) = s^n + \lambda s + 1,$$

where $n = 4m$. Then $P(j\omega, \lambda) = \omega^n + \lambda j\omega + 1$, and Re $P(j\omega, \lambda) \neq 0$ for all λ. Thus, there are no critical values of ω, and the entire real axis is the single root invariant region for the polynomial $P(s, \lambda)$ (indeed, it has $2m$ stable and $2m$ unstable roots for any λ). The minimal number of root invariant regions is one. For the slightly modified example $P(s, k) = \lambda(s^n + 1) + s$, we obtain the real axis with excluded origin as the root invariant region: for any $\lambda \neq 0, P(s, \lambda)$ has $2m$ stable and $2m$ unstable roots.

Theorem 1 can be stated in terms of the Nyquist criterion. We consider a plant with the transfer function $H(s) = \dfrac{a(s)}{b(s)}$ closed by a P-controller with gain k (see Fig. 3).

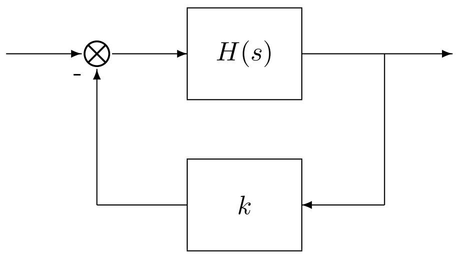

Figure 3: The block-diagram for the feedback system.

The closed-loop characteristic polynomial is $ka(s) + b(s)$. Substituting λ for $-\frac{1}{k}$, we arrive at the polynomial family (4).

Theorem 1'. The Nyquist plot $H(j\omega)$ has no more than $n+1$ intersections μ_i with the real axis. The interval (μ_i, μ_{i+1}) is a stability interval for $\frac{1}{\mu_{i+1}} < k < \frac{1}{\mu_i}$ if

$$2(m_+ - m_-) + p + \delta = 0,$$

where m_+ (m_-) is the number of bottom-up (top-down) intersections of $H(j\omega)$ with the real axis below μ_i, p is the number of unstable roots of $b(s)$, and

$$\delta = \begin{cases} 1, & -\frac{a_0}{b_0} < \mu_i < -\frac{a_n}{b_n}, \\ -1, & -\frac{a_n}{b_n} < \mu_i < -\frac{a_0}{b_0}, \\ 0, & \text{otherwise}. \end{cases}$$

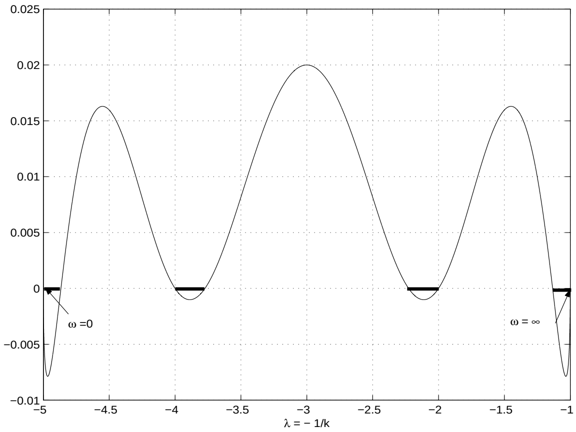

Figure 4: The Nyquist plot

Consider the continuous-time version of Example 1:

$$\begin{aligned} a(s) = & \quad (s+1)^n + \varepsilon(s+1)^{n-2}(s-1)^2 + \alpha(s-1)^n, \\ b(s) = & \quad (s+1)^{n-1}(s-1). \end{aligned}$$

Figure 4 depicts its Nyquist plot for $n = 8$. The system acquires and loses stability $\lfloor \frac{n}{2} \rfloor$ times.

4 One complex parameter

A complex counterpart of the polynomial family (4) is

$$P(s, \lambda) = \{a(s) + \lambda b(s), \ \lambda \in \mathbb{C} \}, \tag{8}$$

where $a(s)$ and $b(s)$ are given real polynomials of degree n.

The curve

$$\lambda(\omega) = -\frac{a_0 + a_1 j\omega + \ldots + a_n (j\omega)^n}{b_0 + b_1 j\omega + \ldots + b_n (j\omega)^n}, \ \omega \in [-\infty, \infty], \tag{9}$$

forms the boundary of the root invariant regions. This curve is bounded when the polynomial $b(s)$ has no roots on the imaginary axis.

Theorem 2 *The polynomial family (8) can have no more than $(n-1)^2 + 2$ root invariant regions.*

Proof. The number of regions depends on the number of self-crossing points of the boundary curve. A bounded curve without self-crossing points divides the parameter plane into two regions, and every simple self-crossing point adds an extra region. Hence at the core of the proof is the need to count up the number of self-crossing points of the algebraic curve (9). Self-crossing points are specified by the condition:

$$\lambda(\omega_1) = \lambda(\omega_2),$$

$$\omega_1 \neq \omega_2.$$

It is equivalent to the system of equations

$$\begin{aligned} \sum_{i=0}^{n-2}\sum_{l=1}^{\frac{n}{2}}(-1)^{i+l}c_{ik}\omega_1^i\omega_2^i(\omega_2^{2l}-\omega_1^{2l}) &= 0 \\ \sum_{i=0}^{n-1}\sum_{l=0}^{\frac{n}{2}}(-1)^{i+l+1}c_{im}\omega_1^i\omega_2^i(\omega_2^{2l+1}-\omega_1^{2l+1}) &= 0, \end{aligned} \tag{10}$$

where $c_{ik} = a_ib_k - a_kb_i,\ k = i+2l,\ m = i+2l+1$.

Now we exploit the following result [1](Appendix), [3].

Theorem (Bezout). Two bivariate polynomials

$$P(x,y) = p_1x^{\alpha_1}y^{\beta_1} + ... + p_kx^{\alpha_k}y^{\beta_k};\ deg(P(x,y)) \doteq \max_i(\alpha_i+\beta_i) = n$$

$$Q(x,y) = q_1x^{\gamma_1}y^{\delta_1} + ... + q_lx^{\gamma_l}y^{\delta_l};\ deg(Q(x,y) = m$$

have no more than mn common zeros.

Due to this theorem, system (10) can have $(2n-2)(2n-1)$ solutions; but this is too crude an estimate. The first equation is identity when $\omega_1+\omega_2 = 0$. This leads to $n-1$ self-crossing points on the real axis λ. Notice that two different solutions (α,β) and (β,α) describe the same self-crossing point. To avoid this degeneracy, we change the variables $\omega \Rightarrow d$:

$$\begin{aligned} \omega_1\omega_2 &= d_1 \\ \omega_1+\omega_2 &= d_2 \neq 0. \end{aligned}$$

For these variables, we have no more than $(n-1)(n-2)$ solutions. The total number of self-crossing points does not exceed $(n-1)+(n-2)(n-1) = (n-1)^2$, and hence the number of root invariant regions is less than or equal to $(n-1)^2+2$. $\diamond$

From the proof, an algebraic algorithm (an extention of Algorithm 1) can be proposed for calculating the number of stable roots in every root invariant region $D(k)$ and for checking the existence and the number of stability regions; however, it is not given here.

Example 3 The uncertain polynomial

$$P(z) = z^n + \lambda z^{n-1} + \alpha;\ \lambda \in \mathbb{C} \tag{11}$$

has $(n-1)^2+1$ root invariant regions for $\alpha > 1$ and 2 root invariant region for $\alpha < 1/(n-1)$. Figure 5 provides the D-decomposition of the complex plane in this example with $n = 6$ and $\alpha = 1.5$.

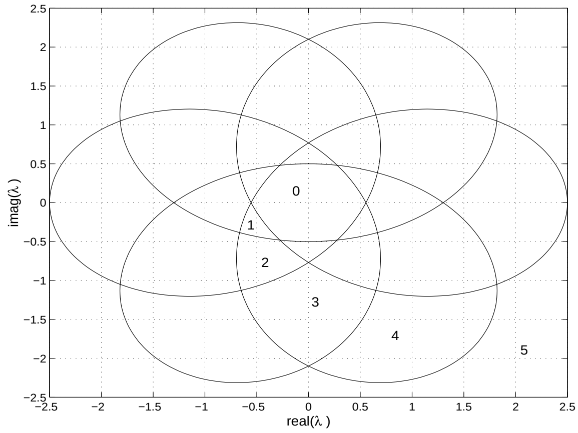

Figure 5: Root invariant regions in Example 3

The D-decomposition is given by the parametric curve

$$\lambda(\omega) = -e^{j\omega} - \alpha e^{-j\omega(n-1)},\ 0 \leq \omega < 2\pi,$$

which describes a hypotrohoid. This curve is generated by a moving point on the complex plane. This motion is a superposition of two rotations. The first rotation has radius 1 and frequency 2π, while the second one has radius α and frequency $(n-1)2\pi$. For $\alpha > 1$, such a curve consists of n arcs, each having two intersections any with other arc. The calculation of the number of intersections gives $n^2 - 2n + 2$; it is just one region less than the upper bound given by Theorem 2.

For $\alpha > 1$ there are no stability regions in Example 3. The issue of the maximal number of stability regions remains unclear.

5 Two real parameters

We now consider the polynomial family with two real parameters:

$$P(s, \lambda_1, \lambda_2) = \{a(s) + \lambda_1 b(s) + \lambda_2 c(s),\ \lambda_1, \lambda_2 \in \mathbb{R}\ \}, \tag{12}$$

where $a(s)$, $b(s)$ and $c(s)$ are given polynomials of degree n with real coefficients.

The structure of D-decomposition is a bit different as compared to the previous section. As usual, D-decomposition contains a complex root boundary curve specified by

$$P(j\omega, \lambda_1, \lambda_2) = 0.$$

Solving this equation with the respect to the parameters gives:

$$\lambda_1 = -\frac{\Delta_1}{\Delta},\quad \lambda_2 = -\frac{\Delta_2}{\Delta}, \tag{13}$$

where

$$\Delta = \begin{vmatrix} U_b & U_c \\ V_b & V_c \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} U_a & U_c \\ V_a & V_c \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} U_b & U_a \\ V_b & V_a \end{vmatrix};$$

$$U_a = a_0 - a_2\omega^2 + a_4\omega^4 + ..., \quad U_b = b_0 - b_2\omega^2 + b_4\omega^4 + ...,$$

$$V_a = a_1 - a_3\omega^2 + a_5\omega^4 + ..., \quad V_b = b_1 - b_3\omega^2 + b_5\omega^4 + ...,$$

$$U_c = c_0 - c_2\omega^2 + c_4\omega^4 + ...$$
$$V_c = c_1 - c_3\omega^2 + c_5\omega^4 + \ldots.$$

This curve begins at the straight line $a_0 + \lambda_1 b_0 + \lambda_2 c_0 = 0$ and ends at the line $a_n + \lambda_1 b_n + \lambda_2 c_n = 0$. These lines are called singular. For a particular ω such that $\Delta = \Delta_1 = \Delta_2 = 0$, we have an extra line rather than a point in the parameter space. All these curves and lines generate the D-decomposition of $\mathbb{R}^2$.

The number of the root invariant regions is still of interest.

Theorem 3 *The polynomial family (12) has no more than* $2n(n-1)+3$ *root invariant regions in the* (λ_1, λ_2) *parameter plane.*

The proof is similar to that of Theorem 2.

The smallest number of root invariant regions is one, see the example below.

Example 4 Let

$$P(s,\lambda) = s^n + \lambda_1 s^3 + \lambda_2 s + 1, n = 4m.$$

Then the equation $P(j\omega,\lambda) = 0$ has no solutions for all ω (because Re $P(j\omega,\lambda) \neq 0$), and the whole $\mathbb{R}^2$ plane is the root invariant region: for any λ, the polynomial $P(s,\lambda)$ has $2m$ stable and $2m$ unstable roots.

This example can be easily extended for any arbitrary number of parameters. Let

$$p(s,k) = a(s^{4m}) + \sum_{i=1}^{r} k_i b_i(s),$$

where $a(t) > 0$ for $t \geq 0$ and $b_i(s)$ are odd polynomials: $b_i(-s) = -b_i(s)$, then $p(s,k)$ has the same number of stable/unstable roots for all $k \in \mathbb{R}^r$. For instance, the polynomial $p(s,k) = 0.1s^{24} + 7.4s^{16} - 13.1s^8 + k_1 s^7 + k_2 s^5 + k_3 s^3 + k_4 s + 15.6$ has entire $\mathbb{R}^4$ as a root invariant region because $a(t) = 0.1t^6 + 7.4t^4 - 13.1t^2 + 15.6 \geq 7.4t^4 - 13.1t^2 + 15.6 > 0$ for $t \geq 0$.

Example 5 The following example demonstrates that the number of root invariant regions is of the order of $O(n^2)$. Let

$$P(s,\lambda) = a(s^2) + s(\lambda_1 b(s^2) + \lambda_2 c(s^2) + \alpha),$$

where $a(t), b(t), c(t)$ are polynomials of degrees $m, m-1, m-1$, respectively (thus $P(s,\lambda)$ has degree $n = 2m$), and $a(t)$ has m negative real roots $-\tau_i^2,\ i = 1,\ldots,m$.

Then the equation of the D-decomposition takes the form $P(j\omega, \lambda) = U(\omega^2) + j\omega V(\omega^2) = 0$ and we obtain the two equations $U(\omega^2) = a(-\omega^2) = 0$, and $\omega V(\omega^2) = \omega(\lambda_1 b(-\omega^2) + \lambda_2 c(-\omega^2) + \alpha) = 0$. The first equation does not depend on λ, it has n real roots $\omega_i = \pm\tau_i$. Hence, D-decomposition is generated by the singular straight lines of the form $\lambda_1 b(\omega_i^2) + \lambda_2 c(\omega_i^2) + \alpha = 0$, and their total number is equal to m. It is seen that m straight lines of generic position divide the plane into $(m^2 + m)/2 + 1$ regions (this well-known fact can be proved by induction), thus $N = n^2/8 + o(n^2)$.

In the particular example below we do not intend to obtain the largest possible number of root invariant regions; we rather aimed at demonstrating how extraordinary D-decomposition can look for polynomials of this type. Let $m = 4$, $a(t) = (1+t)\ (2+t)\dots(8+t)$, $b(t) = (1+t)(3+t)(5+t)(7+t)$, $c(t) = (2+t)(4+t)$ $(6+t)(8+t)$, and $\alpha = 105$. Then we have six orthogonal lines $\lambda_1 = -7$, $\lambda_1 = 1$, $\lambda_1 = 11\frac{2}{3}$, $\lambda_2 = -7$, $\lambda_2 = 1$, $\lambda_2 = 11\frac{2}{3}$ depicted in Fig. 6

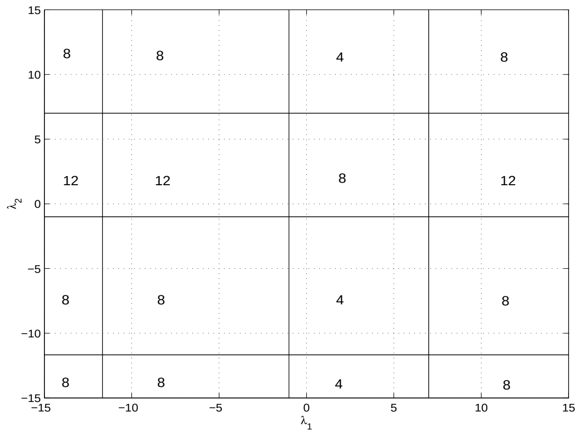

Figure 6: Root invariant regions in Example 5

What is the largest number of stability regions is an open problem. The following example (originated in Nikolayev, 2002) demonstrates that this number can attain $n - 1$.

Example 6 The uncertain polynomial with the a_{n-1}, a_0 coefficients being the parameters

$$P(z) = z^n + a_{n-1}z^{n-1} + (1+\varepsilon)z^{n-2} + a_0, \quad 0 < \varepsilon < 2/(n-2), \tag{14}$$

has $n - 1$ stability regions.

Figures 7 and 8 depict the decomposition of the space of the uncertain parameter into root invariant regions. The behavior of the complex root boundary curve is rather complicated. The curve runs to infinity with some particular values of ω and it has loops. The stability regions are inside these loops.

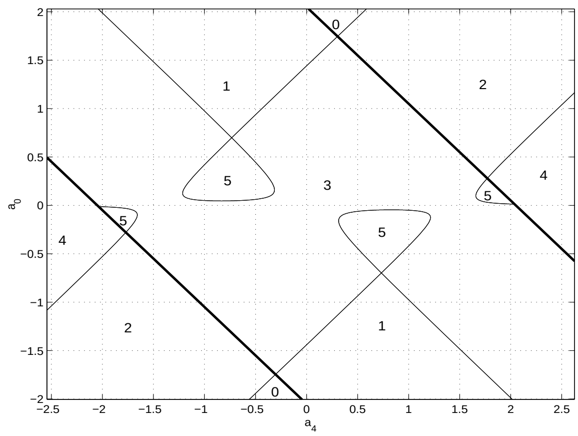

Figure 7: Root invariant regions for $n = 5$ in Example 6

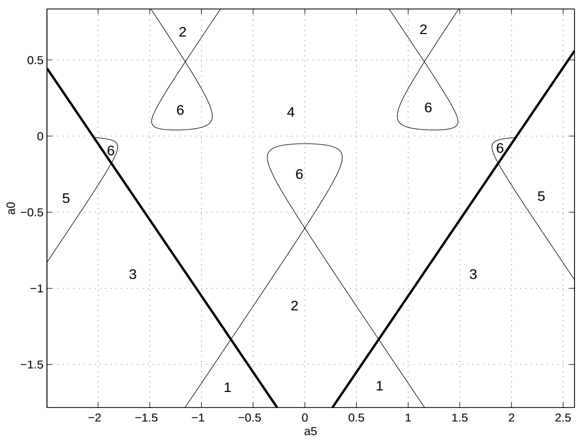

Figure 8: Root invariant regions for $n = 6$ in Example 6

Besides the a_{n-1}, a_0 parameters we have a free parameter ε. We study the behavior of the stability regions as ε increases. The stability regions become smaller and shrink simultaneously for a particular value of ε. This critical value is $\varepsilon^* = \frac{2}{n-2}$, and there is no stability for $\varepsilon > \varepsilon^*$.

6 Conclusions

The root invariant regions geometry in the parameter space can be quite diverse. We prove that for the one-parameter polynomial family, the D-decomposition divides the real axis into no more than $n+1$ segments. Thus, there exist no more than $\lceil \frac{n}{2} \rceil$ stability intervals. In other words, the Nyquist diagram has no more than $n+1$ intersections with the real axis, and there exist no more than $\lceil \frac{n}{2} \rceil$ stability intervals for the gain. We construct an example with the maximal number of the stability intervals and this example has an obvious geometric interpretation. For the case of one complex parameter the maximal possible number of root invariant regions is shown to be $n^2 - 2n + 3$, and this upper bound is tight. Similar results are valid for two real uncertain parameters. We study the discrete-time system of [9] (the parameters are the two coefficients of the polynomial), where the stability domain consists of $n-1$ simply connected regions. In particular, we show that all simply connected regions of the stability domain shrink simultaneously for a particular value of ε. We calculate this critical value and demonstrate that there is no stability for $\varepsilon > \varepsilon^*$.

A lot of related problems on this subject remain open and require further analysis: the case of nonlinear parameter dependence and the stability domain geometry for various $m > 2$, to name just a few.

The results are useful in ow-order controllers design and optimization problems in control theory. An extension of D-decomposition technique to the matrix case is applicable to the construction of the stability domain in the parameter space for systems with scalar gain and double-input double-output systems will be presented in a separate paper.

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